Robust Slater’s Condition in an Uncertain Environment

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Robust Slater’s Condition

Slater’s condition
- a constraint qualification appearing e.g. in optimality conditions in convex optimization
- requires an existence of an interior feasible point

Our goal
- robust Slater’s condition for interval-valued constraints
- we will be content with linear systems

Another motivation
- many is known for interval linear systems ($Ax \leq b$), but less is known for strict systems ($Ax < b$)
Interval matrix

\[ A = \{ A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A} \} \]

The midpoint and the radius matrices:

\[ A_c = \frac{1}{2} (A + \overline{A}), \quad A_\Delta = \frac{1}{2} (\overline{A} - A). \]

System of interval linear inequalities

A family of linear systems

\[ Ax \leq b, \quad A \in A, \quad b \in b \]

In short:

\[ Ax \leq b \]

A realization is any system \( Ax \leq b \), where \( A \in A \) and \( b \in b \).
Intervals, interval vectors, interval matrices, . . .

Example

\[
\begin{pmatrix}
[1, 2] & 3 \\
[-1, 1] & [3, 4]
\end{pmatrix}
\begin{pmatrix}
x
\end{pmatrix}
\leq
\begin{pmatrix}
[11, 12] \\
[5, 6]
\end{pmatrix}
\]

Definition

An interval system \( Ax \leq b \) is

- *strongly solvable* if it is solvable for every realization,

A vector \( x \in \mathbb{R}^n \) is

- *a strong solution* if it solves every realization,

Similarly for other types of systems.
Strong solvability of interval equations

**Theorem**

An interval system \( Ax = b, \ x > 0 \) is strongly solvable iff the system

\[
(A_c + \text{diag}(s)A_\Delta)x = b_c - \text{diag}(s)b_\Delta, \quad x > 0
\]

is solvable for each \( s \in \{\pm 1\}^m \).

**Remark**

- In other words, \((\clubsuit)\) has the form \( Ax = b \), where

\[
A_{i*} = \begin{cases} \overline{A}_{i*} & \text{if } s_i = 1, \\ A_{i*} & \text{if } s_i = -1, \end{cases} \quad b_i = \begin{cases} b_i & \text{if } s_i = 1, \\ \frac{b_i}{b_i} & \text{if } s_i = -1. \end{cases}
\]

- An analogous result as for \( Ax = b, \ x \geq 0 \). [Rohn, 1981]

- Characterization \((\clubsuit)\) is exponential in the number of equations, not variables.

**Theorem**

Checking strong solvability of \( Ax = b, \ x > 0 \) is co-NP-hard.
Theorem

A vector $x$ is a strong solution to an interval system $Ax = b$, $x > 0$, iff

$$A_c x = b_c, \ x > 0, \ A_\Delta = 0, \ b_\Delta = 0.$$

Remark

In other words, all intervals in $Ax = b$ must be degenerate.
Strong solvability of interval inequalities

**Theorem**

An interval system $\mathbf{A} \mathbf{x} < \mathbf{b}$ is strongly solvable iff the system

$$
\overline{\mathbf{A}} \mathbf{x}^1 - \underline{\mathbf{A}} \mathbf{x}^2 < \mathbf{b}, \quad x^1 \geq 0, \quad x^2 \geq 0 \quad \left( \spadesuit \right)
$$

is solvable in variables $x^1, x^2$.

**Remark**

An analogous result as for $\mathbf{A} \mathbf{x} \leq \mathbf{b}$. [Rohn & Kreslová, 1994]

**Theorem**

Let $x^1, x^2$ be a solution to $(\spadesuit)$ and define $\mathbf{x}^* := x^1 - x^2$.

Then $\mathbf{x}^*$ is a solution to $\mathbf{A} \mathbf{x} < \mathbf{b}$ for every $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

**Corollary**

An interval system $\mathbf{A} \mathbf{x} < \mathbf{b}$ is strongly solvable iff it has a strong solution.
Corollary

For a vector $x \in \mathbb{R}^n$, the following conditions are equivalent:

1. $x$ is a strong solution to $Ax < b$,

2. $x = x^1 - x^2$, $\overline{A}x^1 - \underline{A}x^2 < b$, $x^1, x^2 \geq 0$.

3. $A_c x + A_\Delta |x| < b$, 
### Three canonical forms of interval LP

\[
\begin{align*}
\min & \quad c^T x \quad \text{subject to} \quad Ax = b, \ x \geq 0, \\
\min & \quad c^T x \quad \text{subject to} \quad Ax \leq b, \\
\min & \quad c^T x \quad \text{subject to} \quad Ax \leq b, \ x \geq 0.
\end{align*}
\]

**Notation:**

- \( S(A, b, c) \) the optimal solution set for realization \((A, b, c) \in (A, b, c)\).
- The set of all possible optimal solutions

\[
S = \bigcup_{(A, b, c) \in (A, b, c)} S(A, b, c).
\]

### Definition

An interval LP problem is *realization bounded* if \( S(A, b, c) \) is bounded for every realization \((A, b, c) \in (A, b, c)\).
Consequences for interval linear programming

Observation

*If $S$ is bounded, then the interval LP problem is realization bounded.*

*The converse implication does not hold in general.*

Example

Consider the interval LP problem

$$
\min \ x \ \text{subject to} \ [0, 1]x = 1, \ x \geq 0.
$$

- For $a \in (0, 1]$ we have a unique optimal solution $x = 1/a$.
- For $a := 0$ we have an infeasible LP problem.

Thus $S = [1, \infty)$, but the problem is realization bounded.

Open problem

Is the converse implication valid if both primal and dual problems are strongly feasible?
Consequences for interval linear programming

**Theorem (Roos, Terlaky and Vial, 2006)**

For a real LP problem, suppose that primal and dual problems are feasible. Then the optimal solution set is bounded iff the dual problem contains a feasible solution satisfying the inequalities strictly.

**Corollary**

An interval LP problem is realization bounded if for every realization the dual problem contains a feasible solution satisfying the inequalities strictly.
Corollary

1. \( \min\{c^T x : Ax = b, \ x \geq 0\} \) is realization bounded if the system
   \[
   \overline{A}^T y^1 - \underline{A}^T y^2 < c, \ y^1, y^2 \geq 0
   \]
is feasible.

2. \( \min\{c^T x : Ax \leq b\} \) is hard to check since strong solvability of
   \( A^T y = b, \ y < 0 \) is intractable.

3. \( \min\{c^T x : Ax \leq b, \ x \geq 0\} \) is realization bounded if the system
   \( A^T y < c, \ y < 0 \) is feasible.
Example
Reconsider the interval LP problem

$$\min x \text{ subject to } [0, 1]x = 1, \ x \geq 0.$$  
The sufficient condition succeeds.

Example
Consider a variation

$$\min -x \text{ subject to } [0, 1]x = 1, \ x \geq 0;$$  
This problem is also realization bounded, but the sufficient condition fails.
Conclusion

Robust Slater’s condition

- First step to characterize robust Slater’s condition for interval-valued problems.
- Characterization and complexity analysis of strict interval linear systems.

Open problems & challenges

- Computationally cheap sufficient condition for intractable cases.
- Extensions to interval systems of mixed equations and inequalities.
- Under which conditions the realization boundedness implies boundedness of the optimal solution set $S$?
Group on Interval Methods

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