Optimal solution set in interval linear programming and its inner approximation

Milan Hladík

Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic
https://kam.mff.cuni.cz/~hladik/

KOI 2020
The 18th International Conference on Operational Research
Šibenik, Croatia
September 23–25, 2020
Introduction to Interval Analysis

Where interval data do appear

- numerical analysis (handling rounding errors)
  \[ \frac{1}{3} \in [0.33333333333333, 0.33333333333334] \]
  \[ \pi \in [3.1415926535897932384, 3.1415926535897932385] \]

- statistical estimation
  - confidence intervals, prediction intervals (future prices, ...)

- measurement errors
  - fuel consumption, stiffness in truss construction, velocity \((75 \pm 2 \text{ km/h})\)

- discretization
  - time is split in days
  - day range of stock prices – daily min / max

- missing data

Interval notation

A compact interval is \([a, \bar{a}] = [a_c - a_\Delta, a_c + a_\Delta]\) (for matrices entrywise)

We consider deterministic intervals with no distribution on them.
**Interval Linear Programming**

**Interval linear programming**

Family of linear programs \((n \text{ variables}, m \text{ equations})\)

\[
\min c^T x \quad \text{subject to} \quad Ax = b, \; x \geq 0
\]

with \(A \in [A, \overline{A}], \; b \in [b, \overline{b}], \; c \in [c, \overline{c}]\).

- Optimal solution set \(S\): The union of all optimal solutions over all realizations of interval coefficients.
- It is hard to handle \(S\) (characterize, approximate, ...).

**Basis stable case**

The problem is basis stable if a basis \(B\) is optimal for each realization.

- If the problem is basis stable, then \(S\) is a polyhedron described

\[
A_B x_B \leq b, \; \overline{A}_B x_B \geq b, \; x_B \geq 0, \; x_N = 0
\]

and the problem is easy.
Example

Consider an interval linear program

\[
\begin{align*}
\max \ (5, 6, 1, 2) \ T x \quad \text{s.t.} \quad \begin{pmatrix}
-2 & 3 & 7 & 8 \\
6 & 7 & -4 & 5 \\
1 & 1 & 1 & 1
\end{pmatrix} x \leq \begin{pmatrix}
15 & 16 \\
18 & 19 \\
6 & 7
\end{pmatrix}, \ x \geq 0.
\end{align*}
\]

- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area.
Example (Garajová, 2016)

- The optimal solution set may be disconnected and nonconvex.

Consider the interval LP problem

max \( x_2 \) subject to \([-1, 1]x_1 + x_2 \leq 0, \ x_2 \leq 1\).
Our Goal

- So far, attention was paid to outer approximations of $S$

Goal

- Compute an “inner” approximation of $S$ in the form of a box
- Subproblem: Given $k$, find a lower bound for $\max_{x \in S} \{x_k\}$

Our approach

- We propose two methods: local search and genetic algorithm
# Local Search (LS)

## Algorithm LS (based on the derivative of a local approximation)

1. Choose $A, b$ to be the midpoints of the input intervals. Put 
   
   $$c_k = \overline{c}_k, \quad c_i = c_i, \quad i \neq k$$

   While we improve the value of $x_k^*$, perform the following steps:

2. Compute the optimal solution $x^*$ and the optimal basis $B$

3. Put $s = \text{sgn}(A_B^{-1})_{k^*}$ (the sign vector of the $k$th row of $A_B^{-1}$)

4. Put $b := b_c + \text{diag}(s)b_\Delta$

5. Put $A := A_c - \text{diag}(s)A_\Delta$

## Observation

1. Under mild assumptions, each fixed point computed by LS is an optimal solution with maximal $k$th coordinate.

2. If the signs of the entries of $A_B^{-1}$ do not change on $A \in [A, \overline{A}]$, then LS produces an optimal solution with maximal $k$th coordinate in one iteration.
### Genetic Algorithm (GA)

#### Parameters

- **Individuals** = scenarios \((A, b, c)\)
- **Fitness function** = \(k\)th entry of the optimum  
  (so polynomial, but not very cheap to compute)
- **Initial population size**: \(p = mn\)
- **Selection**: 15% of the individuals with the highest fitness value
- **Crossover** of \((A, b, c)\) and \((A', b', c')\):  
  Split \(A\) and \(A'\) into two blocks of rows (of random size) and interchange them. The same approach is applied on \(b, b'\) and \(c, c'\).
- **Mutation**.  
  It is applied on 70% of the population on average. We mutate 30% of the entries to the lower or upper bound of the interval entries.
- **Termination**. The number of iterations is \(2m\).
### Numerical Comparison

basis stable = % of basis stable instances, \( \text{ratio} = \frac{\text{width}(\text{LS})}{\text{width}(\text{GA})} \)

radius = radius of interval coefficients

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>radius</th>
<th>basis stable</th>
<th>ratio</th>
<th>LS (sec.)</th>
<th>GA (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>0.01</td>
<td>0.52</td>
<td>1.00240</td>
<td>0.7592</td>
<td>44.95</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>0.05</td>
<td>0.2</td>
<td>0.97587</td>
<td>0.8481</td>
<td>46.90</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>0.1</td>
<td>0.2</td>
<td>0.92596</td>
<td>0.7652</td>
<td>43.33</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>0.01</td>
<td>0.16</td>
<td>1.29660</td>
<td>1.919</td>
<td>547.5</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>0.05</td>
<td>0</td>
<td>1.31197</td>
<td>2.124</td>
<td>544.7</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>0.1</td>
<td>0</td>
<td>1.14029</td>
<td>2.146</td>
<td>554.0</td>
</tr>
<tr>
<td>100</td>
<td>30</td>
<td>0.01</td>
<td>0</td>
<td>1.64458</td>
<td>7.611</td>
<td>15943</td>
</tr>
<tr>
<td>100</td>
<td>30</td>
<td>0.05</td>
<td>0</td>
<td>1.22515</td>
<td>4.947</td>
<td>16120</td>
</tr>
<tr>
<td>100</td>
<td>30</td>
<td>0.1</td>
<td>0</td>
<td>1.11846</td>
<td>4.235</td>
<td>12423</td>
</tr>
</tbody>
</table>

### Conclusion

LS is mostly tighter and significantly faster than GA.
References


Interval Methods Group

Group on Interval Methods

https://kam.mff.cuni.cz/gim