Two Approaches to Inner Estimations of the Optimal Solution Set in Interval Linear Programming

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Interval Linear Programming

Family of linear programs \((n \text{ variables, } m \text{ equations})\)
\[
\min c^T x \quad \text{subject to} \quad Ax = b, \quad x \geq 0
\]
with \(A \in [A, \overline{A}], b \in [b, \overline{b}], c \in [c, \overline{c}]\).

- The problem is basis stable if a basis \(B\) is optimal for each realization.
- If the problem is basis stable, then \(S\) is a polyhedron described as:
  \[
  \Delta k \geq 0, \quad \overline{A} x \geq \overline{b}, \quad \underline{A} x \geq \underline{b}, \quad x_M = 0
  \]

  and the problem is easy

Goal
- Compute an inner approximation of \(S\) in the form of a box.
- Subproblem: Given \(k\), find a lower bound for \(\max_{x \in S}(x_k)\)

Genetic Algorithm (GA)

Parameters
- Individuals = scenarios \((A, b, c)\)
- Fitness function = \(k\)th entry of the optimum
  (so polynomial, but not very cheap to compute)
- Initial population size: \(p = mn\)
- Selection: 15% of the individuals with the highest fitness value
- Crossover of \((A, b, c)\) and \((\overline{A}, \overline{b}, \overline{c})\):
  Split \(A\) and \(\overline{A}\) into two blocks of rows (of random size) and interchange them. The same approach is applied on \(b, \overline{b}\) and \(c, \overline{c}\).
- Mutation.
  It is applied on for 70% of the population on average. We mutate 30% of the entries to the lower or upper bound of the interval entries.
- Termination. The number of iterations is 2m.

Introduction to Interval Analysis

Where interval data do appear
- numerical analysis (handling rounding errors)
- \(\frac{1}{10^3} \in [0.333333333333, 0.3333333334]\)
- \(\pi \in [3.1415926535897932384, 3.1415926535897932385]\)
- statistical estimation
  - confidence intervals, prediction intervals (future prices, . . .)
- measurement errors
  - fuel consumption, stiffness in truss construction, velocity (75 \pm 2 km/h)
- discretization
  - time is split in days
- day range of stock prices – daily min / max
- missing data

Interval notation
A compact interval is \([a, b]\) = \([a - s\Delta, a + s\Delta]\) (for matrices entrywise)

We consider deterministic intervals with no distribution on them.

Local Search (LS)

Algorithm LS (based on the derivative of a local approximation)
- Choose \(A, b\) to be the midpoints of the input intervals. Put
  \(c_i = \pi_i, \quad c_i = \omega_i, \quad i \neq k\)

  While we improve the value of \(x_k\), perform the following steps:
- Compute the optimal solution \(x^*\) and the optimal basis \(B\)
- Put \(s = \text{sign}(A^*_{ik})x_k\)
- Put \(b := \overline{b} + \text{diag}(s)\Delta\)
- Put \(A := A_i - \text{diag}(s)\Delta\)

Observation
- Under mild assumptions, each fixed point computed by LS is an optimal solution with maximal \(k\)th coordinate.
- If the signs of the entries of \(A^*_{ik}\) do not change on \(A \in [A, \overline{A}]\), then LS produces an optimal solution with maximal \(k\)th coordinate in one iteration

Numerical Comparison

<table>
<thead>
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<th>(m)</th>
<th>(n)</th>
<th>radius</th>
<th>basis stable</th>
<th>ratio (\text{LS (sec.)}/\text{GA (sec.)})</th>
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Conclusion
LS is mostly tighter and significantly faster than GA.

Interval Methods Group

References


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