Two Approaches to Inner Estimations of the Optimal Solution Set in Interval Linear Programming

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Introduction to Interval Analysis

Where interval data do appear

- numerical analysis (handling rounding errors)
  - $\frac{1}{3} \in [0.33333333333333, 0.33333333333334]$
  - $\pi \in [3.1415926535897932384, 3.1415926535897932385]$
- statistical estimation
  - confidence intervals, prediction intervals (future prices, ...)
- measurement errors
  - fuel consumption, stiffness in truss construction, velocity (75 ± 2 km/h)
- discretization
  - time is split in days
  - day range of stock prices – daily min / max
- missing data

Interval notation

A compact interval is $[\underline{a}, \overline{a}] = [a_c - a_\Delta, a_c + a_\Delta]$ (for matrices entrywise)

We consider deterministic intervals with no distribution on them.

Interval Linear Programming

Interval linear programming

Family of linear programs (n variables, m equations)

$$\min\ c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0$$

with $A \in [\underline{A}, \overline{A}]$, $b \in [\underline{b}, \overline{b}]$, $c \in [\underline{c}, \overline{c}]$.

- Optimal solution set S: The union of all optimal solutions over all realizations of interval coefficients.
- The problem is basis stable if a basis B is optimal for each realization.
- If the problem is basis stable, then S is a polyhedron described

$$\underline{A}_B x_B \leq \overline{b}_B,\ \overline{A}_B x_B \geq \underline{b}_B,\ x_B \geq 0,\ x_N = 0$$

and the problem is easy

Goal

- Compute an inner approximation of S in the form of a box.
- Subproblem: Given k, find a lower bound for $\max_{x \in S}\{x_k\}$

Local Search (LS)

Algorithm LS (based on the derivative of a local approximation)

1. Choose A, b to be the midpoints of the input intervals. Put

$$c_k = \overline{c}_k, \quad c_i = \underline{c}_i, \quad i \neq k$$

While we improve the value of x_k^*, perform the following steps:

2. Compute the optimal solution x^* and the optimal basis B
3. Put $s = \operatorname{sgn}(A_B^{-1})_{k*}$ (the sign vector of the kth row of A_B^{-1})
4. Put $b := b_c + \operatorname{diag}(s) b_\Delta$
5. Put $A := A_c - \operatorname{diag}(s) A_\Delta$

Observation

1. *Under mild assumptions, each fixed point computed by LS is an optimal solution with maximal kth coordinate.*
2. *If the signs of the entries of A_B^{-1} do not change on $A \in [\underline{A}, \overline{A}]$, then LS produces an optimal solution with maximal kth coordinate in one iteration*

Genetic Algorithm (GA)

Parameters

- *Individuals* = scenarios (A, b, c)
- *Fitness function* = kth entry of the optimum (so polynomial, but not very cheap to compute)
- *Initial population size:* $p = mn$
- *Selection:* 15% of the individuals with the highest fitness value
- *Crossover* of (A, b, c) and (A', b', c'):
  Split A and A' into two blocks of rows (of random size) and interchange them. The same approach is applied on b, b' and c, c'.
- *Mutation.*
  It is applied on for 70% of the population on average. We mutate 30% of the entries to the lower or upper bound of the interval entries.
- *Termination.* The number of iterations is $2m$.

Numerical Comparison

basis stable = % of basis stable instances, ratio = width(LS)/width(GA)
radius = radius of interval coefficients

m	n	radius	basis stable	ratio	LS (sec.)	GA (sec.)
10	5	0.01	0.52	1.00240	0.7592	44.95
10	5	0.05	0.2	0.97587	0.8481	46.90
10	5	0.1	0.2	0.92596	0.7652	43.33
30	10	0.01	0.16	1.29660	1.919	547.5
30	10	0.05	0	1.31197	2.124	544.7
30	10	0.1	0	1.14029	2.146	554.0
100	30	0.01	0	1.64458	7.611	15943
100	30	0.05	0	1.22515	4.947	16120
100	30	0.1	0	1.11846	4.235	12423

Conclusion

LS is mostly tighter and significantly faster than GA.

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Interval Methods Group

[Group on Interval Methods]

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