Interval robustness of matrix properties for the linear complementarity problem

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LCP with an interval matrix – Introduction

Linear complementarity problem (LCP)

\[ y = Az + q, \quad y, z \geq 0, \]
\[ y^Tz = 0, \]

- LCP appears in many optimization and operations research models (quadratic programming, equilibria in bimatrix games, ...).
- NP-hard to solve

Interval matrix

\[ A := \{ A \in \mathbb{R}^{n \times n} : \underline{A} \leq A \leq \overline{A} \}, \]

where \( \underline{A}, \overline{A} \in \mathbb{R}^{n \times n}, \underline{A} \leq \overline{A}, \) and the inequality is understood entrywise.

The midpoint and the radius matrices are defined as

\[ A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}). \]
Linear complementarity problem (LCP)

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Problem statement

Let \( \mathcal{P} \) be a matrix property. We say that \( \mathcal{P} \) holds strongly for \( \mathbf{A} \) if it holds for each \( A \in \mathbf{A} \).
Motivation

Problem: $Ax = b$.

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonsingular</td>
<td>Unique solution for each $b$</td>
</tr>
<tr>
<td>Full column rank</td>
<td>At most one solution for each $b$</td>
</tr>
<tr>
<td>Full row rank</td>
<td>At least one solution for each $b$</td>
</tr>
</tbody>
</table>
**Motivation**

Problem LCP: \( y = Az + q, \ y^Tz = 0, \ y, z \geq 0. \)

<table>
<thead>
<tr>
<th>matrix type</th>
<th>property</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )-matrix</td>
<td>unique solution for each ( q )</td>
</tr>
<tr>
<td>principally nondegenerate</td>
<td>finitely many solutions (incl. 0) for each ( q )</td>
</tr>
<tr>
<td>strictly copositive</td>
<td>at least one solution for each ( q )</td>
</tr>
<tr>
<td>semimonotone</td>
<td>unique solution for each ( q &gt; 0 )</td>
</tr>
<tr>
<td>column sufficient</td>
<td>the solution set is convex (or empty)</td>
</tr>
<tr>
<td>( R_0 )-matrix</td>
<td>the solution set is bounded</td>
</tr>
<tr>
<td>( R )-matrix</td>
<td>at least one solution for each ( q )</td>
</tr>
</tbody>
</table>
Copositive matrix

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is

- copositive if $x^T A x \geq 0$ for each $x \geq 0$
- strictly copositive if $x^T A x > 0$ for each $x \geq 0$ ($x \geq 0$ and $x \neq 0$)

Properties

- A copositive matrix ensures that the complementary pivot algorithm for solving the LCP works.
- A strictly copositive matrix implies that the LCP has a solution for each $q \in \mathbb{R}^n$.
- Checking whether $A$ is copositive is a co-NP-hard problem (Murty and Kabadi, 1987).
Copositive matrix

**Definition**
A matrix $A \in \mathbb{R}^{n \times n}$ is
- **copositive** if $x^T Ax \geq 0$ for each $x \geq 0$
- **strictly copositive** if $x^T Ax > 0$ for each $x \geq 0$ ($x \geq 0$ and $x \neq 0$)

**Convention**
- $A$ is (strictly) copositive if and only if its symmetric counterpart $\frac{1}{2}(A + A^T)$ is (strictly) copositive.
- That is why we can without loss of generality focus on symmetric matrices.

**Observation**
*A is strongly (strictly) copositive if and only if $A$ is (strictly) copositive.*
Recall

A matrix $A \in \mathbb{R}^{n \times n}$ is

- an $M$-matrix if $A = sl_n - N$ for some $N \geq 0$ such that $s > \rho(N)$,
- an $M_0$-matrix if $A = sl_n - N$ for some $N \geq 0$ such that $s \geq \rho(N)$.

Proposition

Let $A_c$ be an $M$-matrix. Then

1. $A$ is strongly copositive if and only if $A$ is an $M_0$-matrix;
2. $A$ is strongly strictly copositive if and only if $A$ is an $M$-matrix.

Corollary

Let $A_c = I_n$. Then

1. $A$ is strongly copositive if and only if $\rho(A_\Delta) \leq 1$;
2. $A$ is strongly strictly copositive if and only if $\rho(A_\Delta) < 1$. 

Principally nondegenerate matrix

Definition

A is **principally nondegenerate** if all its principal minors are nonzero.

Properties

- The LCP has finitely many solutions (including zero) for every $q \in \mathbb{R}^n$.
- Checking principal nondegeneracy is co-NP-hard (Tseng, 2000).

Proposition

A **is strongly principally nondegenerate if and only if**

$$\det \left( D_{e-|y|} + D_{|y|} A_c D_{|z|} \right) \det \left( D_{e-|y|} + D_{|y|} A_c D_{|z|} - D_y A_{\Delta} D_z \right) > 0$$

for each $y, z \in \{0, \pm 1\}^m$ such that $|y| = |z|$.

It enumerates $5^n$ instances, justified by co-NP-hardness of checking both

- principal nondegeneracy of a real matrix, and
- strong nonsingularity of an interval matrix (Poljak and Rohn, 1993).
Principally nondegenerate matrix

Proposition

Let $A_c$ be an $M$-matrix. Then $A$ is strongly principally nondegenerate if and only if $A$ is an $M$-matrix.

Proposition

Let $A_c$ be positive definite. Then $A$ is strongly principally nondegenerate if and only if it is strongly positive definite.

Remark

Checking strong positive definiteness of $A$ is co-NP-hard, but there are various sufficient conditions known (e.g., Rohn, 1994).
Column sufficient matrix

Definition

A ∈ ℝ^{n×n} is column sufficient matrix if for each pair of disjoint index sets \( I, J \subseteq \{1, \ldots, n\} \), \( I \cup J \neq \emptyset \), the system

\[
\begin{pmatrix}
    A_{I,I} & -A_{I,J} \\
    -A_{J,I} & A_{J,J}
\end{pmatrix}
\begin{pmatrix}
    x_I \\
    x_J
\end{pmatrix}\leq\begin{pmatrix}
    0 \\
    0
\end{pmatrix}, \quad x > 0
\]

is infeasible.

Properties

- For any \( q \in \mathbb{R}^n \) the solution set of the LCP is a convex set (including the empty set).
- Checking column sufficiency is co-NP-hard (Tseng, 2000).
Column sufficient matrix

Proposition

\( \mathbf{A} \) is strongly column sufficient if and only if system

\[
\begin{pmatrix}
\frac{A_{I,I}}{A_{J,I}} & -\frac{A_{I,J}}{A_{J,I}} \\
-\frac{A_{J,I}}{A_{J,J}} & \frac{A_{J,J}}{A_{J,J}}
\end{pmatrix}
\begin{pmatrix}
x \\
x
\end{pmatrix} \preceq 0, \quad x > 0
\]

is infeasible for each admissible \( I, J \).

Proposition

\( \mathbf{A} \) is strongly column sufficient if and only if matrices of the form

\[ A_{ss} = A_c - D_s A_{\Delta} D_s \]

are column sufficient for each \( s \in \{ \pm 1 \}^n \).

Proposition

Let \( A_c \) be an M-matrix and \( A_{\Delta} \) irreducible. Then \( \mathbf{A} \) is strongly column sufficient if and only if \( \mathbf{A} \) is an \( M_0 \)-matrix.

Proposition

Let \( A_c \) be positive semidefinite. Then \( \mathbf{A} \) is strongly column sufficient if and only if it is strongly positive semidefinite.
Example (Convex quadratic programming problem)

\[
\min \ x^T C x + d^T x \quad \text{subject to} \quad B x \leq b, \ x \geq 0.
\]

Optimality conditions for this problem have the form of an LCP

\[
y = A z + q, \quad y^T z = 0, \quad y, z \geq 0,
\]

where

\[
A := \begin{pmatrix} 0 & -B \\ B^T & 2C \end{pmatrix}, \quad q := \begin{pmatrix} b \\ d \end{pmatrix}, \quad z := \begin{pmatrix} u \\ x \end{pmatrix}.
\]

For concreteness, consider the problem

\[
\min \ 10x_1^2 + 8x_1 x_2 + 5x_2^2 + x_1 + x_2 \\
\text{subject to} \quad 2x_1 - x_2 \leq 10, \quad -3x_1 + x_2 \leq 9, \ x \geq 0,
\]

so we have

\[
A = \begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & 3 & -1 \\ 2 & -3 & 20 & 8 \\ -1 & 1 & 8 & 10 \end{pmatrix}.
\]
Example (Convex quadratic programming problem (cont’d))

<table>
<thead>
<tr>
<th>$B_\Delta$</th>
<th>$C_\Delta$</th>
<th>strong properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_\Delta = 0$</td>
<td>$C_\Delta = \frac{1}{4}</td>
<td>C</td>
</tr>
<tr>
<td>$B_\Delta = 0$</td>
<td>$C_\Delta = \frac{1}{3}</td>
<td>C</td>
</tr>
<tr>
<td>$B_\Delta = 0$</td>
<td>$C_\Delta = \frac{9}{10}</td>
<td>C</td>
</tr>
<tr>
<td>$B_\Delta = 0$</td>
<td>$C_\Delta =</td>
<td>C</td>
</tr>
<tr>
<td>$B_\Delta = \frac{1}{10}</td>
<td>B_c</td>
<td>$</td>
</tr>
<tr>
<td>$B_\Delta = \frac{1}{10}</td>
<td>B_c</td>
<td>$</td>
</tr>
<tr>
<td>$B_\Delta = \frac{1}{10}</td>
<td>B_c</td>
<td>$</td>
</tr>
<tr>
<td>$B_\Delta = \frac{1}{5}</td>
<td>B_c</td>
<td>$</td>
</tr>
</tbody>
</table>
References

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