A survey on the interval matrix properties related to the linear complementarity problem

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LCP with an interval matrix – Introduction

Linear complementarity problem (LCP)

$$y = Az + q, \quad y, z \geq 0,$$
$$y^T z = 0,$$

- LCP appears in many optimization and operations research models (quadratic programming, equilibria in bimatrix games, ...).
- NP-hard to solve

Interval matrix

$$\boldsymbol{A} := \{A \in \mathbb{R}^{n\times n} : \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{n\times n}$, $\underline{A} \leq \overline{A}$, and the inequality is understood entrywise.

The midpoint and the radius matrices are defined as

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

LCP with an interval matrix – Introduction

Linear complementarity problem (LCP)

$$
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\end{aligned}
$$

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Problem statement

Let $\mathcal{P}$ be a matrix property. We say that $\mathcal{P}$ holds *strongly* for $\boldsymbol{A}$ if it holds for each $A \in \boldsymbol{A}$.

Motivation

Problem: $Ax = b$.

matrix type	property
nonsingular	unique solution for each b
full column rank	at most one solution for each b
full row rank	at least one solution for each b

LCP with an interval matrix – Motivation

Motivation

Problem LCP: $y = Az + q,\ y^T z = 0,\ y, z \geq 0.$

matrix type	property
P-matrix	unique solution for each q
principally nondegenerate	finitely many solutions (incl. 0) for each q
strictly copositive	at most one solution for each q
semimonotone	unique solution for each $q > 0$
column sufficient	the solution set is convex (or empty)
R_0-matrix	the solution set is bounded
R-matrix	at most one solution for each q

Copositive matrix

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is

- *copositive* if $x^T A x \geq 0$ for each $x \geq 0$
- *strictly copositive* if $x^T A x > 0$ for each $x \gneqq 0$ ($x \geq 0$ and $x \neq 0$)

Properties

- A copositive matrix ensures that the complementary pivot algorithm for solving the LCP works.
- A strictly copositive matrix implies that the LCP has a solution for each $q \in \mathbb{R}^n$.
- Checking whether A is copositive is a co-NP-hard problem (Murty and Kabadi, 1987).

Copositive matrix

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Convention

- A is (strictly) copositive if and only if its symmetric counterpart $\frac{1}{2}(A + A^T)$ is (strictly) copositive.
- That is why we can without loss of generality focus on symmetric matrices.

Observation

$\boldsymbol{A}$ is strongly (strictly) copositive if and only if $\underline{A}$ is (strictly) copositive.

Copositive matrix

Recall

A matrix $A \in \mathbb{R}^{n\times n}$ is

- an M-matrix if $A = sI_n - N$ for some $N \geq 0$ such that $s > \rho(N)$,
- an M_0-matrix if $A = sI_n - N$ for some $N \geq 0$ such that $s \geq \rho(N)$.

Proposition

Let A_c be an M-matrix. Then

(1) $\boldsymbol{A}$ *is strongly copositive if and only if* $\underline{A}$ *is an* M_0*-matrix;*

(2) $\boldsymbol{A}$ *is strongly strictly copositive if and only if* $\underline{A}$ *is an* M*-matrix.*

Corollary

Let $A_c = I_n$. Then

(1) $\boldsymbol{A}$ *is strongly copositive if and only if* $\rho(A_\Delta) \leq 1$*;*

(2) $\boldsymbol{A}$ *is strongly strictly copositive if and only if* $\rho(A_\Delta) < 1$*.*

Copositive matrix – A proof

Proposition

Let A_c be an M-matrix. Then

(1) $\boldsymbol{A}$ *is strongly copositive if and only if* $\underline{A}$ *is an* M_0*-matrix.*

Proof.

"If."

If $\underline{A}$ is an M_0-matrix, then it is positive semidefinite and so is copositive.

"Only if."

- If $\underline{A}$ is not an M_0-matrix, then $A = sI_n - N$, where $N \geq 0$, $\rho(N) > s$.
- For the corresponding Perron vector $x \gneqq 0$ we have $Nx = \rho(N)x \gneqq sx$, from which $Ax \lneqq 0$.
- If $x_i = 0$, then $(Nx)_i = 0$ and so $(Ax)_i = 0$. Similarly, if $x_i > 0$, then $(Nx)_i > sx_i$ and so $(Ax)_i < 0$.
- Hence Ax and x have the same nonzero entries, whence $x^T Ax < 0$; a contradiction. $\square$

Principally nondegenerate matrix

Definition

A is *principally nondegenerate* if all its principal minors are nonzero.

Properties

- The LCP has finitely many solutions (including zero) for every $q \in \mathbb{R}^n$.
- Checking principal nondegeneracy is co-NP-hard (Tseng, 2000).

Proposition

A *is strongly principally nondegenerate if and only if*

$$\det\left(D_{e-|y|} + D_{|y|}A_c D_{|z|}\right)\det\left(D_{e-|y|} + D_{|y|}A_c D_{|z|} - D_y A_\Delta D_z\right) > 0$$

for each $y, z \in \{0, \pm 1\}^m$ *such that* $|y| = |z|$.

It enumerates 5^n instances, justified by co-NP-hardness of checking both

- principal nondegeneracy of a real matrix, and
- strong nonsingularity of an interval matrix (Poljak and Rohn, 1993).

Principally nondegenerate matrix

Proposition

Let A_c be an M-matrix. Then $\boldsymbol{A}$ is strongly principally nondegenerate if and only if $\underline{A}$ is an M-matrix.

Proposition

Let A_c be positive definite. Then $\boldsymbol{A}$ is strongly principally nondegenerate if and only if it is strongly positive definite.

Remark

Checking strong positive definiteness of $\boldsymbol{A}$ is co-NP-hard, but there are various sufficient conditions known (e.g., Rohn, 1994).

Column sufficient matrix

Definition

$A \in \mathbb{R}^{n\times n}$ is *column sufficient matrix* if for each pair of disjoint index sets $I, J \subseteq \{1, \dots, n\}$, $I \cup J \neq \emptyset$, the system

$$\begin{pmatrix} A_{I,I} & -A_{I,J} \\ -A_{J,I} & A_{J,J} \end{pmatrix} x \lneqq 0, \quad x > 0$$

is infeasible.

Properties

- For any $q \in \mathbb{R}^n$ the solution set of the LCP is a convex set (including the empty set).
- Checking column sufficiency is co-NP-hard (Tseng, 2000).

Column sufficient matrix

Proposition

$\boldsymbol{A}$ is strongly column sufficient if and only if system

$$\begin{pmatrix} \underline{A}_{I,I} & -\overline{A}_{I,J} \\ -\overline{A}_{J,I} & \underline{A}_{J,J} \end{pmatrix} x \lneqq 0, \quad x > 0 \quad \textit{is infeasible for each admissible } I, J.$$

Proposition

$\boldsymbol{A}$ is strongly column sufficient if and only if matrices of the form $A_{ss} = A_c - D_s A_\Delta D_s$ are column sufficient for each $s \in \{\pm 1\}^n$.

Proposition

Let A_c be an M-matrix and A_Δ irreducible. Then $\boldsymbol{A}$ is strongly column sufficient if and only $\underline{A}$ is an M_0-matrix.

Proposition

Let A_c be positive semidefinite. Then $\boldsymbol{A}$ is strongly column sufficient if and only if it is strongly positive semidefinite.

R_0-matrix

Definition

$A \in \mathbb{R}^{n \times n}$ is an R_0-*matrix* if the LCP with $q = 0$ has the only solution $x = 0$.
Equivalently, for each index set $\emptyset \neq I \subseteq \{1, \dots, n\}$, the system

$$A_{I,I}x = 0, \quad A_{J,I}x \geq 0, \quad x > 0$$

is infeasible, where $J := \{1, \dots, n\} \setminus I$.

Properties

- For any $q \in \mathbb{R}^n$ the LCP has a bounded solution set.
- Checking R_0-matrix property is co-NP-hard (Tseng, 2000).

R_0-matrix

Proposition

$\boldsymbol{A}$ is strongly R_0-matrix if and only if system

$$\underline{A}_{I,I}x \le 0, \quad \overline{A}_{I,I}x \ge 0, \quad \underline{A}_{J,I}x \ge 0, \quad x > 0$$

is infeasible for each admissible I, J.

Proposition

Let A_c be an M-matrix. Then $\boldsymbol{A}$ is strongly an R_0-matrix if and only if $\underline{A}$ is an M-matrix.

Corollary

Let $A_c = I_n$. Then $\boldsymbol{A}$ is strongly an R_0-matrix if and only if $\rho(A_\Delta) < 1$.

Example

Example (Convex quadratic programming problem)

$$\min \; x^T Cx + d^T x \;\; \text{subject to} \;\; Bx \le b, \; x \ge 0.$$

Optimality conditions for this problem have the form of an LCP

$$y = Az + q, \quad y^T z = 0, \quad y, z \ge 0,$$

where

$$A := \begin{pmatrix} 0 & -B \\ B^T & 2C \end{pmatrix}, \quad q := \begin{pmatrix} b \\ d \end{pmatrix}, \quad z := \begin{pmatrix} u \\ x \end{pmatrix}.$$

For concreteness, consider the problem

$$\begin{aligned} \min \;\; & 10x_1^2 + 8x_1x_2 + 5x_2^2 + x_1 + x_2 \\ \text{subject to} \;\; & 2x_1 - x_2 \le 10, \; -3x_1 + x_2 \le 9, \; x \ge 0, \end{aligned}$$

so we have

$$A = \begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & 3 & -1 \\ 2 & -3 & 20 & 8 \\ -1 & 1 & 8 & 10 \end{pmatrix}.$$

Example

Example (Convex quadratic programming problem (cont'd))

B_Δ	C_Δ	strong properties
$B_\Delta = 0$	$C_\Delta = \frac{1}{4}\lvert C\rvert$	semimonotone, column sufficient R-matrix, R_0-matrix
$B_\Delta = 0$	$C_\Delta = \frac{1}{3}\lvert C\rvert$	semimonotone, R-matrix, R_0-matrix
$B_\Delta = 0$	$C_\Delta = \frac{9}{10}\lvert C\rvert$	semimonotone, R-matrix, R_0-matrix
$B_\Delta = 0$	$C_\Delta = \lvert C\rvert$	semimonotone
$B_\Delta = \frac{1}{10}\lvert B_c\rvert$	$C_\Delta = \frac{1}{10}\lvert C\rvert$	semimonotone, column sufficient R-matrix, R_0-matrix
$B_\Delta = \frac{1}{10}\lvert B_c\rvert$	$C_\Delta = \frac{1}{5}\lvert C\rvert$	semimonotone, column sufficient R-matrix, R_0-matrix
$B_\Delta = \frac{1}{10}\lvert B_c\rvert$	$C_\Delta = \frac{1}{2}\lvert C\rvert$	semimonotone, R-matrix, R_0-matrix
$B_\Delta = \frac{1}{5}\lvert B_c\rvert$	$C_\Delta = \frac{1}{5}\lvert C\rvert$	$\emptyset$

Conclusion

Conclusion

- matrix properties of the linear complementarity problem extended to interval matrices
- challenging: new polynomial cases

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