Radius of Stability of Different Matrix Properties Related to Optimization Problems

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Outline

Matrix properties

- positive definiteness (relates to convexity of a function)
- P-matrix property (unique solvability of LCP)
- M-matrix property (Leontief's input-output model)
- H-matrix property
- total positivity
- inverse nonnegativity

Problem statement

Given $A \in \mathbb{R}^{n \times n}$, determine the radius of stability of a matrix property for a matrix norm (= distance to a nearest violated matrix).

Matrix norms

Vector p-norms: $\|x\|_p := \left(\sum_{i=1}^n |x|_i^p\right)^{\frac{1}{p}}$, $p \geq 1$.

Particular matrix norms

- The subordinate matrix norm

$$\|A\|_{\alpha,\beta} := \max_{\|x\|_\alpha = 1} \|Ax\|_\beta$$

- The induced p-norm

$$\|A\|_p := \max_{\|x\|_p = 1} \|Ax\|_p$$

- Spectral norm (induced 2-norm)

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \sigma_{\max}(A).$$

- Frobenius norm $\quad \|A\|_F := \sqrt{\sum_{i,j} a_{ij}^2}$
- max-norm $\quad \|A\|_{\max} := \max_{i,j} |a_{ij}| = \|A\|_{1,\infty}$

Matrix norms

Properties of matrix norms

(P1) consistent norm: if $\|AB\| \leq \|A\| \cdot \|B\|$ for every $A, B \in \mathbb{R}^{n \times n}$
(for induced norms, Frobenius, but not for max-norm)

(P2) $\|I_n\| = 1$
(for induced norms and max-norm, not for Frobenius)

(P3) $\|A'\| \leq \|A\|$ whenever A' is a submatrix of A
(for induced p-norms, Frobenius and max-norm)

(P4) $\|e_i e_j^T\| = 1 \ \ \forall i, j$
(for induced p-norms, Frobenius and max-norm)

Regularity radius

Definition

Regularity radius of $A \in \mathbb{R}^{n\times n}$ is the distance to the nearest singular matrix

$$\mathrm{r}(A) := \min\{\|A - B\| : B \text{ is singular}\}.$$

Particular cases

- For the spectral, Frobenius and some orthogonally invariant norms,

$$\mathrm{r}(A) = \sigma_{\min}(A)$$

- For any induced matrix norm (Gastinel–Kahan theorem),

$$\mathrm{r}(A) = \|A^{-1}\|^{-1}$$

- For the max-norm,

$$\mathrm{r}(A) = \|A^{-1}\|_{\infty,1}^{-1} = \frac{1}{\max_{y,z\in\{\pm 1\}^n} y^T A^{-1} z}$$

Its computation is NP-hard [Poljak and Rohn, 1993]

SDP approximation [Hartman and Hladík, 2016]

Positive definiteness

Definition

Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Radius of positive definiteness of A is

$$\delta^* := \sup\{\delta \geq 0 : A + A' \text{ is positive definite } \forall A' : A' = A'^T,\ \|A'\| < \delta\}.$$

Theorem

For every consistent matrix norm satisfying (P2) *(i.e.,* $\|I_n\| = 1$*) we have* $\delta^* = \lambda_{\min}(A)$*, the smallest eigenvalue of* A*.*

For max-norm

- co-NP-hard to check $\delta^* > 1$,
- $\delta^* \geq \frac{1}{n}\lambda_{\min}(A)$,
- $\delta^* = \min_{y\in\{\pm 1\}^n} \frac{1}{y^T A^{-1} y}$,
- If A is inverse nonnegative, then $\delta^* = \frac{1}{e^T A^{-1} e}$.

P-matrix property

Definition

$A \in \mathbb{R}^{n\times n}$ is a P-matrix if all its principal minors are positive.

- It guarantees a unique solution for each q of the LCP

$$q + Ax \geq 0, \quad x \geq 0, \quad (q + Ax)^T x = 0$$

[Cottle, Pang, and Stone, 2009; Murty, 1988]

- Checking P-matrix property is co-NP-hard [Coxson, 1994]
- Efficiently recognizable subclasses:
  - positive definite matrices,
  - M-matrices,
  - H-matrices with positive diagonal,
  - or totally positive matrices.

P-matrix radius of a P-matrix A

$$\delta^* := \sup\{\delta \geq 0 : A + A' \text{ is an P-matrix } \forall A' : \|A'\| < \delta\}.$$

P-matrix property

Theorem

For any matrix norm we have

$$\delta^* = \min\{\mathrm{r}(\hat{A}) : \hat{A} \text{ is a principal submatrix of } A\}.$$

In particular, for the spectral or Frobenius norm we have

$$\delta^* = \min\{\sigma_{\min}(\hat{A}) : \hat{A} \text{ is a principal submatrix of } A\}.$$

Theorem

Suppose A is a symmetric positive definite or an M-matrix ($a_{ij} \leq 0$, $i \neq j$, and $A^{-1} \geq 0$). For the spectral or Frobenius norm we have

$$\delta^* = \sigma_{\min}(A).$$

Theorem

Suppose A is an M-matrix. For the max-norm we have

$$\delta^* = \frac{1}{e^T A^{-1} e}.$$

M-matrix property

Definition

$A \in \mathbb{R}^{n\times n}$ is an M-matrix if $a_{ij} \leq 0$ for every $i \neq j$ and $A^{-1} \geq 0$ (or, $Av > 0$ for certain $v > 0$). [Horn and Johnson, 1991]

- sub-class of P-matrices
- stability of Leontief's input-output analysis in economic systems, and others

M-matrix radius of an M-matrix A

$$\delta^* := \sup\{\delta \geq 0 : A + A' \text{ is an M-matrix } \forall A' : \|A'\| < \delta\}.$$

Example

Consider the identity matrix $A = I_n$ and the spectral norm:

- the P-matrix radius is 1
- the M-matrix radius is 0

M-matrix property

Theorem

For every matrix norm satisfying (P3) *and* (P4) *we have*

$$\delta^* = \min_{i \neq j}\{-a_{ij}, \mathrm{r}(A)\}.$$

In particular, for the spectral or Frobenius norm, we have

$$\delta^* = \min_{i \neq j}\{-a_{ij}, \sigma_{\min}(A)\}.$$

Max-norm

- The worst case is $A - \delta E$, where E consists of ones.
- δ^* is maximal such that $A - \delta E$ is an M-matrix for all $\delta \in [0, \delta^*)$.
- Simple parametrization (linear constraints by Sherman–Morrison formula):

$$(A - \delta E)_{ij} \leq 0, \;\; i \neq j, \;\; \text{and } (A - \delta E)^{-1} \geq 0.$$

Total positivity

Definition

$A \in \mathbb{R}^{n\times n}$ is totally positive if the determinants of all submatrices are positive.

- Sub-class of P-matrices.
- Only initial submatrices $A^{(1)}, \ldots, A^{(n^2)}$ needed to check: rows are indexed by $\{1, \ldots, k\}$ and columns by $\{\ell, \ldots, \ell + k - 1\}$ or vice versa.
  [Fallat and Johnson, 2011]

Totally positive radius of A

$$\delta^* := \sup\{\delta \geq 0 : A + A' \text{ is totally positive } \forall A' : \|A'\| < \delta\}.$$

Total positivity

Theorem

We have $\delta^* = \min_{i=1,\dots,n^2} r(A^{(i)})$.

In particular, for the spectral or Frobenius norm, $\delta^* = \min_{i=1,\dots,n^2} \sigma_{\min}(A^{(i)})$.

Max-norm

- The worst case is $A - \delta ss^T$ or $A + \delta ss^T$, where $s := (1, -1, 1, -1, \dots)^T$ [Garloff, 1982]
- δ^* is thus computed by simple parametrization (Sherman–Morrison formula)

Inverse nonnegativity

Definition

$A \in \mathbb{R}^{n\times n}$ is inverse nonnegative if $A^{-1} \geq 0$.

Inverse nonnegativity radius of A

$$\delta^* := \sup\{\delta \geq 0 \colon A + A' \text{ is inverse nonnegative } \forall A' : \|A'\| < \delta\}.$$

Theorem

We have $\delta^* = \min_{i,j=1,\dots,n}\{r(A), r(A^{ij})\}$.
In particular, for the spectral or Frobenius norm,
$\delta^* = \min_{i,j=1,\dots,n}\{\sigma_{\min}(A), \sigma_{\min}(A^{ij})\}$.

Max-norm

- The worst case is $A - \delta E$ or $A + \delta E$ [Kuttler, 1971]
- δ^* is thus computed by simple parametrization (Sherman–Morrison formula)

Conclusion

Conclusion

- stability radius for diverse matrix properties related to optimization
- typically reduced to several problems of regularity radius
- often for many norms tractable (spectral of Frobenius norm), sometimes NP-hard (max-norm)

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