Approximation of the Solution Set of Absolute Value Equations

Milan Hladík

Department of Applied Mathematics
Faculty of Mathematics and Physics,
Charles University, Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

EUROPT 2019
The 17th Workshop on Advances in Continuous Optimization,
Glasgow, UK
June 28-29, 2019

AVE – Introduction

Absolute value equation (AVE)

$$Ax - b = B|x|,$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

Properties

- This problem is equivalent to the linear complementarity problem.
- Solving AVE is an NP-hard problem (Mangasarian, 2007).
- Checking uniqueness of a solution of AVE is NP-hard, too (Prokopyev, 2009).
- Diverse iterative methods were proposed (incl. generalized Newton and smoothing techniques).

AVE – Introduction

Absolute value equation (AVE)

$$Ax - b = B|x|,$$

where $A, B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.

Goal

Develop outer approximation techniques for the solutions of AVE:

- A tight outer approximation of the solutions in the form of a convex polyhedral set
- If the outer approximation set is empty, this will also serve as a condition for unsolvability.
- A sufficient condition for the existence of 2^n solutions lying in mutually different orthants.

AVE – Introduction

Absolute value equation (AVE)

$$Ax - b = B|x|,$$

where $A, B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.

Approximation and consequences

Let $\mathcal{S} \subset \mathbb{R}^n$ be an enclosure of the solution set to AVE.

- If $\mathcal{S}$ lies in one orthant, then the problem is easily resolved.

  (Let $s \in \{\pm 1\}^n$ be the sign vector. Then AVE reads $(A - B\,\mathrm{diag}(s))x = b$, which is a standard linear equation system.)

AVE – Polyhedral enclosures

Properties

- Initial bounds: Let u be such that $|x| \leq u$ for each solution x.
- Let us split $B = B^+ - B^-$ into the positive and negative parts $B^+, B^- \geq 0$. Now, AVE $Ax - b = B|x|$ reads

$$Ax - b + B^-|x| = B^+|x|$$

- The inequality $Ax - b + B^-|x| \leq B^+|x|$ is relaxed as follows

$$Ax - b + B^- y \leq B^+ u, \quad \pm x \leq y$$

- The converse inequality $Ax - b + B^-|x| \geq B^+|x|$ is relaxed as

$$Ax - b + B^- u \geq B^+ y, \quad \pm x \leq y$$

- Polyhedral outer estimation

$$Ax + B^- y \leq b + B^+ u, \quad -Ax + B^+ y \leq -b + B^- u, \quad \pm x - y \leq 0$$

AVE of type I

Absolute value equation of type I

$$x - b = B|x|$$

(easily obtained provided A is nonsingular)

Properties

- Uniqueness of the solution under the condition $\rho(B^T B) < 1$ (Mangasarian and Meyer, 2006).
- Uniqueness of the solution under the condition $\rho(|B|) < 1$ (Rohn et al, 2014).

AVE of type I – Interval enclosures

Absolute value equation of type I

$$x - b = B|x|$$

(easily obtained provided A is nonsingular)

Proposition (Bauer–Skeel type bounds)

If $\rho(|B|) < 1$, then the solution x satisfies

$$|x - b| \leq (I - |B|)^{-1}|B||b|.$$

Proposition (Hansen–Bliek–Rohn type bounds)

If $\rho(|B|) < 1$, then the solution x satisfies $x \in \boldsymbol{x}^{HBR}$, where

$$\boldsymbol{x}_i^{HBR} = \frac{b_i + (u_i/d_i - |b|_i))[-1, 1]}{1 + (1 - 1/d_i)[-1, 1]}, \quad i = 1, \ldots, n,$$

where $C = I - |B|$, $u = C^{-1}|b|$ and $d_i = (C^{-1})_{ii}$.

Numerical experiments – orthants

Example

α … norm of matrix B

`unique` … portion of instances uniquely determining the orthant

`ort. num.` … number of orthants intersected by the enclosure

			Bauer–Skeel		polyhedral	
n	α	runs	unique	ort. num.	unique	ort. num.
10	0.1	10^4	0.8303	1.193	0.9957	1.005
10	0.2	10^4	0.6636	1.447	0.9857	1.014
10	0.5	10^4	0.2696	2.731	0.8827	1.128
10	0.75	10^4	0.0756	5.168	0.7158	1.372
100	0.1	10^2	0.12	7.80	0.94	1.06
100	0.2	10^2	0.00	72.4	0.87	1.15
100	0.5	10^2	0.00	10^5	0.15	5.79
100	0.75	10^2	0.00	10^8	0.01	58.45

Numerical experiments – times and tightness

Example

α . . . norm of matrix B

`tightness` . . . average sum of radii of the enclosure w.r.t. the polyhedral enclosure

		Bauer–Skeel		polyhedral	
n	α	time (s)	tightness	time (s)	tightness
10	0.1	0.0003358	51.48	0.3733	1
10	0.2	0.0003200	25.19	0.3530	1
10	0.5	0.0003116	9.583	0.3347	1
10	0.75	0.0003101	6.108	0.3168	1
100	0.1	0.01658	44.78	76.77	1
100	0.2	0.01585	21.90	72.05	1
100	0.5	0.01503	8.160	67.95	1
100	0.75	0.01491	5.149	68.47	1

AVE of type II – Existence of 2^n solutions

Absolute value equation of type II

$$Ax - b = |x|$$

Example: AVE $e = |x|$ has 2^n solutions: every vector in $\{\pm 1\}^n$.

Proposition (Mangasarian & Meyer, 2006)

There are 2^n solutions, lying in interiors of mutually different orthants, if

$$b < 0 \quad \textit{and} \quad \|A\|_\infty < \frac{\min_i\{|b|_i\}}{2\max_i\{|b|_i\}}.$$

Proposition (more general than the above)

Let $\rho(|A|) < 1$ and $b < 0$. There are 2^n solutions if

(i) $2|b| > G(I - |A|)^{-1}|b|$, *or*

(ii) $|b| > 2|A||b|$,

where $G = \mathrm{diag}\left(1/(I - |A|)^{-1}_{11}, \ldots, 1/(I - |A|)^{-1}_{nn}\right)$.

AVE of type II – Non-existence tests

Absolute value equation of type II

$$Ax - b = |x|$$

Corollary

AVE has no solution if

$$\rho(|A|) < 1 \ \ \textit{and} \ \ x_\Delta = -(I - |A|)^{-1} b \ \textit{is not nonnegative} \tag{1}$$

Proposition (Mangasarian & Meyer, 2006)

AVE has no solution if the system is feasible

$$r \geq A^T r \geq -r, \quad b^T r > 0 \tag{2}$$

Proposition

If $A \geq 0$ is irreducible, then we have

(i) (1) *implies solvability of* (2),

(ii) *Solvability of* (2) *implies* (1) *with the condition in the form* $\rho(A) \leq 1$.

Conclusion

- outer approximation for the solutions of AVE (polyhedral, interval box)
- a sufficient condition for unsolvability
- a sufficient condition for the existence of 2^n solutions lying in mutually different orthants
- the approximation is often able to determine the signs of the solution(s)

References

M. Hladík.
Bounds for the solutions of absolute value equations.
Comput. Optim. Appl., 69(1):243–266, 2018.

O. L. Mangasarian and R. R. Meyer.
Absolute value equations.
Linear Algebra Appl., 419(2):359–367, 2006.

O. A. Prokopyev.
On equivalent reformulations for absolute value equations.
Comput. Optim. Appl., 44(3):363–372, 2009.

J. Rohn.
A Manual of Results on Interval Linear Problems.
Technical Report 1164, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2012.
http://www.library.sk/arl-cav/en/detail/?&idx=cav_un_epca*0381706

J. Rohn, V. Hooshyarbakhsh and R. Farhadsefat.
An iterative method for solving absolute value equations and sufficient conditions for unique solvability.
Optim. Lett., 8(1):35–44, 2014.