Several tests for recognizing pseudoconvexity on a restricted domain

Milan Hladík* Lubomir V. Kolev Iwona Skalna

Department of Applied Mathematics
Faculty of Mathematics and Physics,
Charles University, Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

EURO 2019
30th European Conference on Operational Research, Dublin, Ireland
June 23–26, 2019

Pseudoconvexity

Motivation

Convexity has many nice properties in the context of optimization.
What about its generalizations?

Definition

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and $\mathcal{S} \subset \mathbb{R}^n$ an open convex set.
Then $f(x)$ is *pseudoconvex* on $\mathcal{S}$ if for every $x, y \in \mathcal{S}$ we have

$$\nabla f(x)^T(y - x) \geq 0 \quad \Rightarrow \quad f(y) \geq f(x).$$

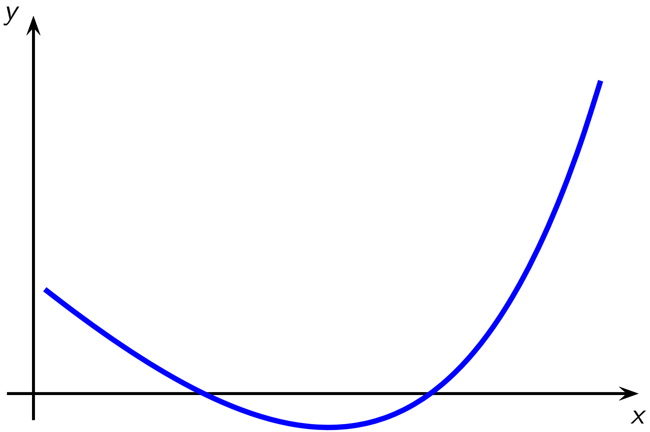
Key Properties

Minimizing pseudoconvex objective functions on convex feasible sets,

- each stationary point is a global minimum,
- each local minimum is a global minimum,
- the optimal solution set is convex.

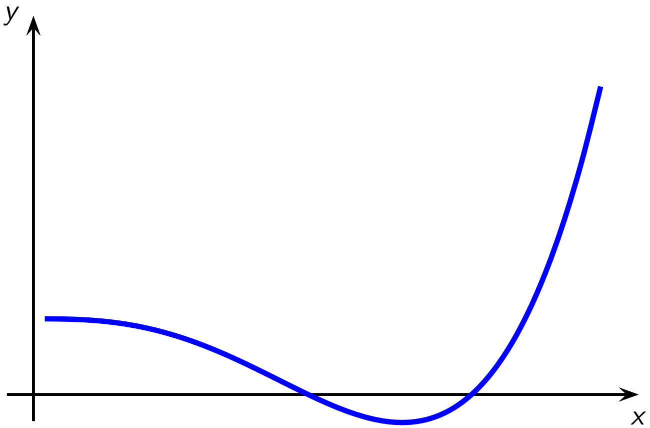
Illustration

Convex function



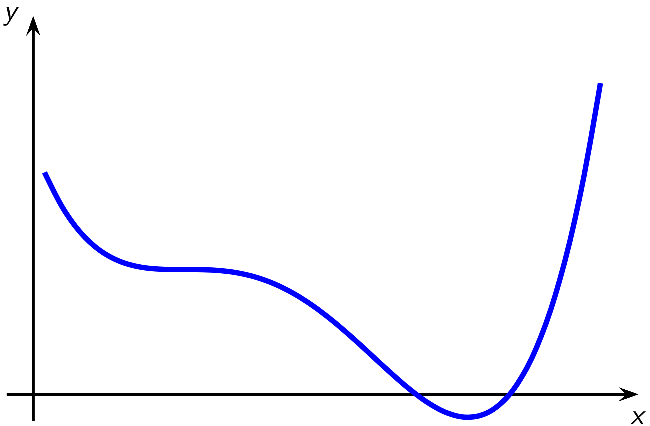
Illustration

Pseudoconvex function



Illustration

Quasiconvex function



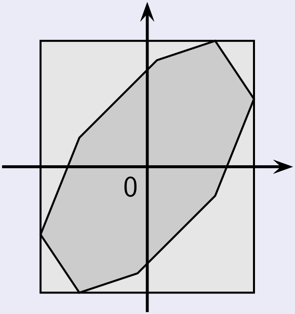
Problem Formulation

Problem formulation

Given an affine form

$$\boldsymbol{x}(\boldsymbol{p}) := \left\{ \sum_{k=1}^{K} x^{(k)} p_k + x_c,\ \ p_k \in \boldsymbol{p}_k \equiv [-1, 1] \right\}.$$

Geometrically it is a zonotope



The question

Is a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ pseudoconvex on $\boldsymbol{x}(\boldsymbol{p})$?

Problem Formulation

Theorem (Ahmadi et al., 2013)

Deciding pseudoconvexity is NP-hard on a class of quartic polynomials.

Aim

Therefore we will be content with cheap sufficient conditions.

Technical Tools

Interval analysis

Interval arithmetic:

$$
\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \\
\boldsymbol{x} - \boldsymbol{y} &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \\
\boldsymbol{x}\boldsymbol{y} &= [\min(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}), \max(\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y})], \\
\boldsymbol{x}/\boldsymbol{y} &= [\min(\underline{x}/\underline{y}, \underline{x}/\overline{y}, \overline{x}/\underline{y}, \overline{x}/\overline{y}), \max(\underline{x}/\underline{y}, \underline{x}/\overline{y}, \overline{x}/\underline{y}, \overline{x}/\overline{y})], \quad 0 \notin \boldsymbol{y}.
\end{aligned}
$$

Evaluation of functions and their derivatives,...

For interval matrix $\boldsymbol{A}$: regularity, eigenvalues, det, positive semidef., ...

Technical Tools

Affine arithmetic (reduced/revised version)

Given two affine forms

$$\boldsymbol{x}(p) := \sum_{k=1}^{K} x_k p_k + \boldsymbol{x}_0 = x^T p + \boldsymbol{x}_0,$$
$$\boldsymbol{y}(p) := \sum_{k=1}^{K} y_k p_k + \boldsymbol{y}_0 = y^T p + \boldsymbol{y}_0,$$

where $p \in \boldsymbol{p}$. For any $\alpha, \beta \in \mathbb{R}$ we have

$$\boldsymbol{x}(p) + \boldsymbol{y}(p) = (x + y)^T p + (\boldsymbol{x}_0 + \boldsymbol{y}_0),$$
$$\alpha \boldsymbol{x}(p) = (\alpha x)^T p + (\alpha \boldsymbol{x}_0).$$

Nonlinear operations have to be approximated. Multiplication usually reads

$$\boldsymbol{x}(p) \cdot \boldsymbol{y}(p) = ((y_0)_c x + (x_0)_c y)^T p + \boldsymbol{z},$$

where $\boldsymbol{z} = [z_c - z_\Delta, z_c + z_\Delta]$ encloses the accumulative error with

$$z_c = x_c y_c + \tfrac{1}{2} x^T y,$$
$$z_\Delta = |x_c| y_\Delta + |y_c| x_\Delta + \left(|x|^T e + x_\Delta\right)\left(|y|^T e + y_\Delta\right) - \tfrac{1}{2}|x|^T |y|.$$

Method Based on Mereau and Paquet

Theorem (Mereau and Paquet, 1974)

The function $f(x)$ is pseudoconvex on set $\mathcal{S}$ if there is $\alpha \geq 0$ such that

$$\nabla^2 f(x) + \alpha \nabla f(x) \nabla f(x)^T \tag{$\star$}$$

is positive semidefinite for all $x \in \mathcal{S}$.

Proposition

We have that $(\star)$ is positive semidefinite if and only if

$$\begin{pmatrix} -\frac{1}{\alpha} & \nabla f(x)^T \\ \nabla f(x) & \nabla^2 f(x) \end{pmatrix}$$

has at most one simple negative eigenvalue.

Method Based on Mereau and Paquet

Theorem (Mereau and Paquet, 1974)

The function $f(x)$ is pseudoconvex on set $\mathcal{S}$ if there is $\alpha \geq 0$ such that

$$\nabla^2 f(x) + \alpha \nabla f(x) \nabla f(x)^T \tag{$\star$}$$

is positive semidefinite for all $x \in \mathcal{S}$.

Method

- Enclose $\nabla f(\boldsymbol{x}(\boldsymbol{p})) \subseteq \boldsymbol{g}(\boldsymbol{p})$, $\nabla^2 f(\boldsymbol{x}(\boldsymbol{p})) \subseteq \boldsymbol{H}(\boldsymbol{p})$
- Denote

$$\boldsymbol{D}'(\boldsymbol{p}) := \begin{pmatrix} -\frac{1}{\alpha} & \boldsymbol{g}(\boldsymbol{p})^T \\ \boldsymbol{g}(\boldsymbol{p}) & \boldsymbol{H}(\boldsymbol{p}) \end{pmatrix} \supseteq \begin{pmatrix} -\frac{1}{\alpha} & \nabla f(x)^T \\ \nabla f(x) & \nabla^2 f(x) \end{pmatrix}$$

- Check that the second smallest eigenvalue of the matrices $\boldsymbol{D}'(\boldsymbol{p})$ stays nonnegative.

Method Based on Crouzeix

Theorem (Crouzeix, 1998)

Function $f(x)$ is pseudoconvex on $\mathcal{S}$ if for each $x \in \mathcal{S}$ and every $y \neq 0$ such that $\nabla f(x)^T y = 0$ we have $y^T \nabla^2 f(x) y > 0$.

Equivalently, by Crouzeix (1998),

$$D(x) := \begin{pmatrix} 0 & \nabla f(x)^T \\ \nabla f(x) & \nabla^2 f(x) \end{pmatrix}.$$

has n positive eigenvalues on $\mathcal{S}$.

Method

- Compute
$$\boldsymbol{D}(\boldsymbol{p}) := \begin{pmatrix} 0 & \boldsymbol{g}(\boldsymbol{p})^T \\ \boldsymbol{g}(\boldsymbol{p}) & \boldsymbol{H}(\boldsymbol{p}) \end{pmatrix} \supseteq \begin{pmatrix} 0 & \nabla f(x)^T \\ \nabla f(x) & \nabla^2 f(x) \end{pmatrix}$$
- Compute an enclosure $\boldsymbol{\lambda}_2$ for the second smallest eigenvalue of $\boldsymbol{D}(\boldsymbol{p})$ and check that $\underline{\lambda}_2 > 0$.

Method Based on Crouzeix and Ferland

Theorem (Crouzeix and Ferland, 1982)

Function $f(x)$ is pseudoconvex on $\mathcal{S}$ if for each $x \in \mathcal{S}$ either $\nabla^2 f(x)$ is positive semidefinite, or $\nabla^2 f(x)$ has one simple negative eigenvalue and there is $b \in \mathbb{R}^n$ such that $\nabla^2 f(x) b = \nabla f(x)$ and $\nabla f(x)^T b < 0$.

Preliminaries

- Enclose $\nabla f(\boldsymbol{x}(\boldsymbol{p})) \subseteq \boldsymbol{g}(\boldsymbol{p})$, $\nabla^2 f(\boldsymbol{x}(\boldsymbol{p})) \subseteq \boldsymbol{H}(\boldsymbol{p})$
- Condition

$$\exists b: \;\; Hb = g, \;\; g^T b < 0$$

is equivalent to $g^T H^{-1} g < 0$ for each $g \in \boldsymbol{g}(\boldsymbol{p})$ and $H \in \boldsymbol{H}(\boldsymbol{p})$.

The method checks that

– every matrix in $\boldsymbol{H}(\boldsymbol{p})$ has at most one simple negative eigenvalue,

– we have $g^T H^{-1} g < 0$ for every $g \in \boldsymbol{g}(p)$, $H \in \boldsymbol{H}(p)$ and $p \in \boldsymbol{p}$.

Method Based on Crouzeix and Ferland

The method checks that

– every matrix in $\boldsymbol{H}(\boldsymbol{p})$ has at most one simple negative eigenvalue, (1)

– we have $g^T H^{-1} g < 0$ for every $g \in \boldsymbol{g}(p)$, $H \in \boldsymbol{H}(p)$ and $p \in \boldsymbol{p}$. (2)

Proposition

- *It is NP-hard to check* (1)*.*
- *It is NP-hard to check* (2) *even for functions of type* $f(x) = \frac{1}{2}x^T A x$*.*

Two tests for (1)

- Compute an enclosing interval $\boldsymbol{\lambda}_2$ for the second smallest eigenvalue of the matrices in $\boldsymbol{H}(\boldsymbol{p})$ and then check whether $\underline{\lambda}_2 \geq 0$.
- Check for $\lambda_2(H_c) > 0$ and regularity of $\boldsymbol{H}(\boldsymbol{p})$.

Method Based on Crouzeix and Ferland

The method checks that

– every matrix in $\boldsymbol{H}(\boldsymbol{p})$ has at most one simple negative eigenvalue, (1)

– we have $g^T H^{-1} g < 0$ for every $g \in \boldsymbol{g}(p)$, $H \in \boldsymbol{H}(p)$ and $p \in \boldsymbol{p}$. (2)

Proposition

- *It is NP-hard to check* (1).
- *It is NP-hard to check* (2) *even for functions of type* $f(x) = \frac{1}{2}x^T A x$.

Two tests for (2)

- Let $\boldsymbol{y}(\boldsymbol{p})$ be an affine form enclosure of the solution set of $\boldsymbol{H}(\boldsymbol{p})y = \boldsymbol{g}(\boldsymbol{p})$. Check that $\boldsymbol{g}(p)^T \boldsymbol{y}(p) < 0$.
- Check that $\det(D_c) < 0$ and $\boldsymbol{D}(\boldsymbol{p})$ is regular.

Numerical Experiments

Example (Random choices of $\boldsymbol{H}(\boldsymbol{p})$ and $\boldsymbol{g}(\boldsymbol{p})$)

$n =$ dimension, $d =$ radius of intervals, $K =$ number of parameters.

Success rate (in %):

			standard interval approach					parametric approach						
n	K	d	MP1	MP2	F	CF	C	MPa	MPb	CF1a	CF1b	CF2a	CF2b	C
5	5	0.1	21.0	35.0	**36.0**	**36.0**	**36.0**	36.0	**40.0**	0.0	0.0	**40.0**	18.0	36.0
10	5	0.1	7.0	30.0	**37.0**	**37.0**	27.0	27.0	**58.0**	0.0	0.0	53.0	28.0	27.0
5	5	0.3	0.0	18.0	**30.0**	**30.0**	20.0	20.0	**39.0**	0.0	0.0	**39.0**	14.0	20.0
10	5	0.3	0.0	5.0	9.0	**10.0**	3.0	3.0	**52.0**	0.0	0.0	39.0	20.0	3.0
5	10	0.1	7.0	18.0	18.0	**24.0**	19.0	19.0	**30.0**	0.0	0.0	**30.0**	9.0	19.0
10	10	0.1	0.0	19.0	**24.0**	**24.0**	13.0	13.0	**61.0**	0.0	0.0	51.0	29.0	13.0
15	10	0.1	0.0	5.0	**7.0**	**7.0**	2.0	2.0	**61.0**	0.0	0.0	36.0	23.0	2.0
5	10	0.3	0.0	4.0	12.0	**18.0**	8.0	8.0	**31.0**	0.0	0.0	**31.0**	14.0	8.0
10	10	0.3	0.0	0.0	0.0	0.0	**1.0**	1.0	**20.0**	0.0	0.0	11.0	4.0	1.0
15	10	0.3	0.0	0.0	0.0	0.0	0.0	0.0	**8.0**	0.0	0.0	2.0	1.0	0.0
5	10	0.5	0.0	0.0	0.0	**4.0**	0.0	0.0	**13.0**	0.0	0.0	**13.0**	4.0	0.0
10	10	0.5	0.0	0.0	0.0	0.0	0.0	0.0	**1.0**	0.0	0.0	**1.0**	**1.0**	0.0

Parametric methods approx. 7 times slower.

Numerical Experiments

Example (Benchmark data (success rate in %))

function	n	div	conv	standard interval approach MP1	MP2	F	CF	C	parametric approach MPa	MPb	CF1a	CF1b	CF2a	CF2b	C
mhw4d	5	5	0.0	0	0	0	0	0	0	**100**	0	0	0	**100**	0
mhw4d	5	8	0.0	0	0	0	0	0	0	**100**	0	0	**100**	**100**	0
Colville	4	5	0.0	0	0	0	0	0	0	0	0	0	0	0	0
Colville	4	10	0.0	0	0	0.2	**0.3**	0	0	**6.6**	0	0	0	**6.6**	0
Rosenbrock	4	5	0.0	0	0	0.2	**0.3**	0	0	0	0	0	0	0	0
Rosenbrock	4	10	0.0	0	0	0.2	**0.3**	0	0	**13.4**	0	0	6.8	**13.4**	0
Rosenbrock	4	15	0.0	0	0	**2.3**	1.8	0	0	**35.5**	0	0	19.6	35.5	0
G&P	2	10	0.0	0	0	0	0	0	0	0	0	0	0	0	0
G&P	2	50	0.4	0	0	0	0	**0.1**	1.9	**2.8**	0	0	**2.8**	2.6	1.9
G&P	2	100	1.4	0	0	0	0	**1.1**	3.4	**7.1**	0	0	**7.1**	4.7	3.4
f5_Messine	3	10	0.0	0	0	0	0	0	0	0	0	0	0	0	0
f5_Messine	3	30	0.0	0	0	0.1	0.2	**1.5**	**3.9**	15.7	10.0	13.9	10.5	**18.8**	4.0
f5_Messine	3	40	0.0	0	0	3.2	4.4	**4.6**	**6.7**	17.9	14.2	20.6	14.3	**24.8**	7.2
6hum_camel	2	10	0.0	0	0	0	0	0	0	0	0	0	0	0	0
6hum_camel	2	50	31.1	4.1	11.3	32.1	**36.9**	26.6	35.0	**56.2**	2.1	2.1	**56.2**	53.2	35.0
6hum_camel	2	100	43.1	10.9	16.7	45.2	**50.8**	40.9	48.6	**62.2**	5.2	5.2	**62.2**	60.1	48.6

Conclusion

Summary

- Compared to direct interval approach, parametric methods are significantly more efficient, but slightly slower.

Next steps?

- Pseudoconvex envelopes instead of convex envelopes?
- Some variation of the αBB method with pseudoconvex functions?

References

J. Crouzeix and J. A. Ferland.
Criteria for quasi-convexity and pseudo-convexity: Relationships and comparisons.
Math. Program., 23(1):193–205, 1982.

J.-P. Crouzeix.
Characterizations of Generalized Convexity and Generalized Monotonicity, A Survey.
In J.-P. Crouzeix et al., eds., *Generalized Convexity, Generalized Monotonicity: Recent Results*, pages 237–256. Springer, 1998.

J. A. Ferland.
Mathematical programming problems with quasi-convex objective functions.
Math. Program., 3(1):296–301, 1972.

P. Mereau and J.-G. Paquet.
Second order conditions for pseudo-convex functions.
SIAM J. Appl. Math., 27:131–137, 1974.