Parametric Solution Methods for Parametric Systems of Equations

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SCAN 2018, Tokyo, Japan,
September 10–15

Introduction

Interval matrix

An interval matrix

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A^c := \tfrac{1}{2}(\overline{A} + \underline{A}), \quad A^\Delta := \tfrac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $m \times n$ interval matrices: $\mathbb{IR}^{m\times n}$.

Introduction

Parametric interval system

Consider a parametric interval linear system

$$A(p)x = b(p),$$

in which parameters have a linear structure

$$A(p) = \sum_{k=1}^{K} A^{(k)} p_k, \quad b(p) = \sum_{k=1}^{K} b^{(k)} p_k.$$

Herein,

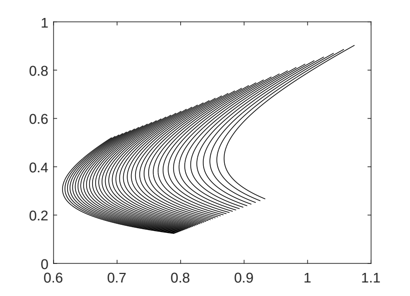
- $A^{(1)}, \ldots, A^{(K)} \in \mathbb{R}^{n \times n}$ and $b^{(1)}, \ldots, b^{(K)} \in \mathbb{R}^n$ are fixed,
- parameters $p_1, \ldots, p_K$ come from interval domains $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K \in \mathbb{IR}$.

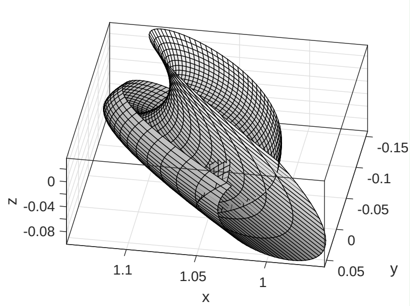
Solution set

$$\Sigma = \{x \in \mathbb{R}^n;\ \exists p \in \boldsymbol{p} : A(p)x = b(p)\}.$$

Examples of Solution Sets

Example





$$\begin{pmatrix} p_2 & 1+2p_1 \\ 3p_2 & -3p_2 \end{pmatrix} x = \begin{pmatrix} 2p_2 \\ 1 \end{pmatrix},$$

$$p_1, p_2 \in [0.6, 2.05]$$

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_1 & 2 & p_1 \\ p_2 & p_1 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ p_1^2 \\ p_2^2 \end{pmatrix},$$

$$p_1 \in [0.0, 1.0], \;\; p_2 \in [0.0, 0.9],$$

Objective

Traditional approach

Find a tight box (an interval vector) containing Σ.

Drawback. Often poor approximation of the set.

p-solution (Kolev, 2014, 2016; Skalna & H., 2017)

Find an enclosure in the form of a zonotope

$$x(p) = A(p)^{-1}b(p) \in Lp + \boldsymbol{x}, \quad p \in \boldsymbol{p}$$

for some $L \in \mathbb{R}^{n \times K}$ and $\boldsymbol{x} \in \mathbb{IR}^n$.

Advantages.

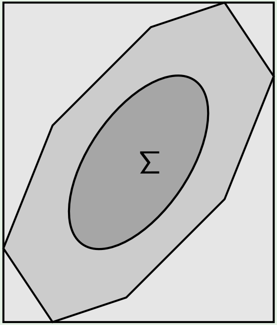
- If needed, box enclosure is computed simply as $L\boldsymbol{p} + \boldsymbol{x}$
- We also have an inner estimation

$$\left[Lp^c - |L|p^\Delta + \overline{x},\ Lp^c + |L|p^\Delta + \underline{x}\right] \subseteq \mathrm{hull}(\Sigma)$$

- Suitable for further processing (in optimization, CSP,...)

Box vs. *p*-solution

Example



- Interval box and a finer enclosure by a zonotope.
- Zonotopes are special convex symmetric polyhedra (images of boxes under linear mappings).

Interval Tools

Traditional tool – interval arithmetic

$$\boldsymbol{a} + \boldsymbol{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$
$$\boldsymbol{a} - \boldsymbol{b} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}],$$
$$\boldsymbol{a} \cdot \boldsymbol{b} = [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})],$$
$$\boldsymbol{a} / \boldsymbol{b} = [\min(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}), \max(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b})], \quad 0 \notin \boldsymbol{b}.$$

Affine arithmetic

Affine form one-dimensional parameter

$$\hat{\boldsymbol{x}}(p) := x^T p + \boldsymbol{x}, \quad p \in \boldsymbol{p} = [-1, 1]^K.$$

Addition and multiples by $\alpha \in \mathbb{R}$:

$$\hat{\boldsymbol{x}}(p) + \hat{\boldsymbol{y}}(p) := (x + y)^T p + (\boldsymbol{x} + \boldsymbol{y}), \quad p \in \boldsymbol{p}$$
$$\alpha\hat{\boldsymbol{x}}(p) := (\alpha x)^T p + (\alpha \boldsymbol{x}), \quad p \in \boldsymbol{p}.$$

Nonlinear operations, including *multiplication*, must be approximated, e.g.,

$$\hat{\boldsymbol{x}}(p) \cdot \hat{\boldsymbol{y}}(p) := (y^c x + x^c y)^T p + \boldsymbol{z},$$

where $\boldsymbol{z}$ encloses the accumulative error set.

(Optimal $\boldsymbol{z}$ can be computed in $\mathcal{O}(n)$ by Skalna & H., 2017.)

Some of Methods

Iterative Methods

- Constraint satisfaction technique (Kolev, 2014)
- A class of iterative methods (Kolev, 2016)
- Gauss–Seidel type approach (Skalna & H., 2017)
- Krawczyk type approach (Skalna & H., 2018)

Direct Methods

- Parametric direct method (Kolev, 2016)
- Generalized expansion method (Skalna & H., 2018)

Preliminaries

Assume affine form of the interval parametric system

$$\hat{\boldsymbol{A}}(p)x = \hat{\boldsymbol{b}}(p), \quad p \in \boldsymbol{p} = [-1, 1]^K$$

where

$$\hat{\boldsymbol{A}}(p) = \textstyle\sum_{k=1}^K A^{(k)} p_k + \boldsymbol{A}, \quad p \in \boldsymbol{p},$$
$$\hat{\boldsymbol{b}}(p) = \textstyle\sum_{k=1}^K b^{(k)} p_k + \boldsymbol{b}, \quad p \in \boldsymbol{p}.$$

(Nonlinear dependencies are linearized by affine arithmetic.)

Preconditioning

Assume preconditioning by midpoint inverse such that the system reads

$$\hat{\boldsymbol{A}}(p)x = \hat{\boldsymbol{b}}(p), \quad p \in \boldsymbol{p}$$

with $A^c \approx I_n$.

Residual correction

Shift x such that $b^c = 0$. $(x \mapsto x - (A^c)^{-1} b^c)$

Iterative Methods: Krawczyk-Type Iterations

Krawczyk-Type Iterations

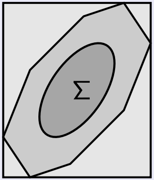
$$\hat{\boldsymbol{x}}(p) \mapsto \hat{\boldsymbol{b}}(p) + (I_n - \hat{\boldsymbol{A}}(p))\hat{\boldsymbol{x}}(p),$$

where $\hat{\boldsymbol{x}}(p)$ is a p-enclosure of the solution set, and the right-hand side is evaluated by affine arithmetic.

Proposition

If $\rho(\sum_{k=1}^{K} |A^{(k)}| + A^{\Delta}) < 1$, then

- *the iterations converge to a unique fixed point for each initial $\boldsymbol{x}(p)$,*
- *its interval hull equals the Parametric Bauer–Skeel enclosure.*



Generalized Expansion Method

Expansion of the inverse matrix

Let $A \in \hat{\boldsymbol{A}}(p)$, $p \in \boldsymbol{p}$, and denote $B := I_n - A$.

If $\rho(B) < 1$, then by Neumann series

$$A^{-1} = (I_n - B)^{-1} = \sum_{i=0}^{\infty} B^i = \sum_{i=0}^{m} B^i + A^{-1}B^{m+1}$$
$$\subseteq \sum_{i=0}^{m} B^i + \boldsymbol{H}B^{m+1},$$

where $\boldsymbol{H}$ is computed as follows:

- denote $\boldsymbol{C} := \hat{\boldsymbol{A}}(\boldsymbol{p})$, for which $C^c = I_n$,
- then $\boldsymbol{H} = \mathrm{hull}\{C^{-1},\ C \in \boldsymbol{C}\}$ is effectively computable (Rohn, 2011):

$$\boldsymbol{H} = [-M + \mathrm{diag}(z), M], \quad M := \underline{C}^{-1} \geq 0, \quad z_i := \frac{2M_{ii}^2}{2M_{ii} - 1}.$$

Generalized Expansion Method

The resulting p-solution computed by affine arithmetic (2 versions)

$$\hat{\boldsymbol{x}}(p) := \left(\sum_{i=0}^{m} \hat{\boldsymbol{B}}(p)^i + \boldsymbol{H}\hat{\boldsymbol{B}}(p)^{m+1} \right) \hat{\boldsymbol{b}}(p)$$

$$:= \sum_{i=0}^{m} \hat{\boldsymbol{B}}(p)^i \hat{\boldsymbol{b}}(p) + \boldsymbol{H}\hat{\boldsymbol{B}}(p)^{m+1} \hat{\boldsymbol{b}}(p).$$

- The second one faster, but not always tighter.
- Practically, $m = 3$ seems to be a good choice.
- Provably as good as the Parametric direct method (Kolev, 2016), which is the case with $m = -1$.

Numerical Results and Conclusion

- Competitive to most of the common methods.
- Useful for both standard and parametric (linear or nonlinear) interval equations.
- Benefit of the p-solution affine form: smaller enclosing set, inner estimation, useful for further processing.

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