Interval Robustness in Linear Programming

Milan Hladík

Department of Applied Mathematics
Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

ROBUST 2018
Rybník, January 21–26, 2018

Outline

Next Section

Interval Data – Motivation

What is interval computation

Solving problems with interval data
(or using interval techniques for non-interval problems)

Important notice

We consider intervals in a set sense, no distribution, no fuzzy shape.

Interval paradigm

Take into account all possible realizations rigorously.

Interval Data

Where interval data do appear

- numerical analysis (handling rounding errors)
  - $\frac{1}{3} \in [0.33333333333333, 0.33333333333334]$
  - $\pi \in [3.1415926535897932384, 3.1415926535897932385]$.
- constraint solving and global optimization
  - find robot singularities, where it may breakdown
    check joint angles $[0, 180]^\circ$
  - find minimum of $f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))$
- statistical estimation
  - confidence intervals, prediction intervals (future prices,...)
- measurement errors
  - fuel consumption, stiffness in truss construction, velocity (75 ± 2 km/h)
- discretization
  - time is split in days
  - day range of stock prices – daily min / max
- missing data

Interval Matrices

Definition (Interval matrix)

An interval matrix is the family of matrices

$$\boldsymbol{A} = \{A \in \mathbb{R}^{m\times n} : \underline{A} \le A \le \overline{A}\},$$

The midpoint and the radius matrices are defined as

$$A_c := \tfrac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \tfrac{1}{2}(\overline{A} - \underline{A}).$$

The set of all interval $m \times n$ matrices is denoted by $\mathbb{IR}^{m\times n}$.

Basic problem

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{IR}^n$. Determine the image

$$f(\boldsymbol{x}) = \{f(x) : x \in \boldsymbol{x}\},$$

or at least its tight interval enclosure.

Interval Arithmetic

Interval Arithmetic (proper rounding used when implemented)

For arithmetical operations $(+, -, \cdot, \div)$, their images are readily computed

$$\begin{aligned}
\boldsymbol{a} + \boldsymbol{b} &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
\boldsymbol{a} - \boldsymbol{b} &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
\boldsymbol{a} \cdot \boldsymbol{b} &= [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})], \\
\boldsymbol{a} \div \boldsymbol{b} &= [\min(\underline{a} \div \underline{b}, \underline{a} \div \overline{b}, \overline{a} \div \underline{b}, \overline{a} \div \overline{b}), \max(\underline{a} \div \underline{b}, \underline{a} \div \overline{b}, \overline{a} \div \underline{b}, \overline{a} \div \overline{b})].
\end{aligned}$$

Some basic functions $\boldsymbol{x}^2$, $\exp(\boldsymbol{x})$, $\sin(\boldsymbol{x})$, ..., too.

Can we evaluate every arithmetical expression on intervals?
Yes, but with overestimation in general due to dependencies.

Example (Evaluate $f(x) = x^2 - x$ on $\boldsymbol{x} = [-1, 2]$)

$$\begin{aligned}
\boldsymbol{x}^2 - \boldsymbol{x} &= [-1, 2]^2 - [-1, 2] = [-2, 5], \\
\boldsymbol{x}(\boldsymbol{x} - 1) &= [-1, 2]([-1, 2] - 1) = [-4, 2], \\
(\boldsymbol{x} - \tfrac{1}{2})^2 - \tfrac{1}{4} &= ([-1, 2] - \tfrac{1}{2})^2 - \tfrac{1}{4} = [-\tfrac{1}{4}, 2].
\end{aligned}$$

Software

Matlab/Octave libraries

- *Intlab* (by S.M. Rump),
  interval arithmetic and elementary functions
  `http://www.ti3.tu-harburg.de/~rump/intlab/`
- *Interval* package for Octave (by O. Heimlich),
  free package of verified interval functions
  `https://wiki.octave.org/Interval_package`
- *Lime* (by M. Hladík, J. Horáček et al.),
  interval methods written in Intlab, under development
  `http://kam.mff.cuni.cz/~horacek/projekty/lime/`

Other languages libraries

- *Int4Sci Toolbox* (by Coprin team, INRIA),
  A Scilab Interface for Interval Analysis
  `http://www-sop.inria.fr/coprin/logiciels/Int4Sci/`
- *C++ libraries*: C-XSC, PROFIL/BIAS, BOOST interval, FILIB++,...
- *many others*: for Fortran, Pascal, Lisp, Maple, Mathematica,...

References – books

G. Alefeld and J. Herzberger.
Introduction to Interval Computations.
Academic Press, New York, 1983.

L. Jaulin, M. Kieffer, O. Didrit, and É. Walter.
Applied Interval Analysis.
Springer, London, 2001.

R. E. Moore.
Interval Analysis.
Prentice-Hall, Englewood Cliffs, NJ, 1966.

R. E. Moore, R. B. Kearfott, and M. J. Cloud.
Introduction to Interval Analysis.
SIAM, Philadelphia, PA, 2009.

A. Neumaier.
Interval Methods for Systems of Equations.
Cambridge University Press, Cambridge, 1990.

J. Rohn.
A handbook of results on interval linear problems.
Tech. Rep. 1163, Acad. of Sci. of the Czech Republic, Prague, 2012.
http://uivtx.cs.cas.cz/~rohn/publist/!aahandbook.pdf

Interval Linear Equations

Interval linear equations

Let $\boldsymbol{A} \in \mathbb{IR}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{IR}^{m}$. The family of systems

$$Ax = b, \quad A \in \boldsymbol{A},\ b \in \boldsymbol{b}.$$

is called interval linear equations and abbreviated as $\boldsymbol{A}x = \boldsymbol{b}$.

Solution set

The solution set is defined

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \boldsymbol{A} \exists b \in \boldsymbol{b} : Ax = b\}.$$

Important notice

We do not want to compute $\boldsymbol{x} \in \mathbb{IR}^n$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Theorem (Oettli–Prager, 1964)

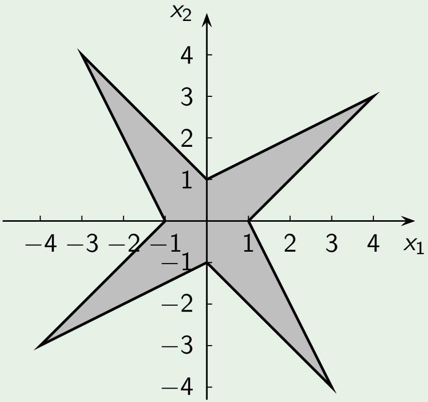
The solution set Σ is a non-convex polyhedral set described by

$$|A_c x - b_c| \leq A_\Delta |x| + b_\Delta.$$

Interval Linear Equations

Example (Barth & Nuding, 1974))

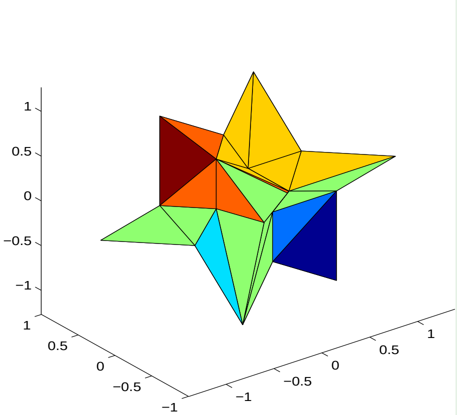
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$



Example of the Solution Set

Example

$$\begin{pmatrix} [3,5] & [1,3] & -[0,2] \\ -[0,2] & [3,5] & [0,2] \\ [0,2] & -[0,2] & [3,5] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}.$$



Topology of the Solution Set

Proposition

In each orthant, Σ is either empty or a convex polyhedral set.

Proof.

Restriction to the orthant given by $s \in \{\pm 1\}^n$:

$$|A_c x - b_c| \leq A_\Delta |x| + b_\Delta, \ \operatorname{diag}(s)x \geq 0.$$

Since $|x| = \operatorname{diag}(s)x$, we have

$$|A_c x - b_c| \leq A_\Delta \operatorname{diag}(s)x + b_\Delta, \ \operatorname{diag}(s)x \geq 0.$$

Using $|a| \leq b \ \Leftrightarrow a \leq b, \ -a \leq b$, we get

$$(A_c - A_\Delta \operatorname{diag}(s))x \leq \overline{b}, \ (-A_c - A_\Delta \operatorname{diag}(s))x \leq -\underline{b}, \ \operatorname{diag}(s)x \geq 0.$$ □

Corollary

The solutions of $\boldsymbol{A}x = \boldsymbol{b}$, $x \geq 0$ *is described by* $\underline{A}x \leq \overline{b}, \ \overline{A}x \geq \underline{b}, \ x \geq 0.$

Topology of the Solution Set

Theorem (Jansson, 1997)

When $\Sigma \neq \emptyset$, then exactly one of the following alternatives holds true:

1. *Σ is bounded and connected ($\boldsymbol{A}$ is regular).*
2. *Each topologically connected component of Σ is unbounded ($\boldsymbol{A}$ is irregular).*

Remark

Checking $\Sigma \neq \emptyset$ and boundedness are NP-hard.

Two basic polynomial cases

1. $A_c = I_n$,
2. $\boldsymbol{A}$ is inverse nonnegative, i.e., $A^{-1} \geq 0 \ \forall A \in \boldsymbol{A}$.

Theorem (Kuttler, 1971)

$\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is inverse nonnegative if and only if $\underline{A}^{-1} \geq 0$ and $\overline{A}^{-1} \geq 0$.

(Then $\square\Sigma = [\overline{A}^{-1}\underline{b}, \underline{A}^{-1}\overline{b}]$ when $\underline{b} \geq 0$, etc.)

Interval Linear Equations

Enclosure

Since Σ is hard to determine and deal with, we seek for enclosures

$$\boldsymbol{x} \in \mathbb{IR}^n \text{ such that } \Sigma \subseteq \boldsymbol{x}.$$

Many methods exist (GE,. . .), usually employ preconditioning.

Preconditioning (Hansen, 1965)

Let $R \in \mathbb{R}^{n \times n}$. The preconditioned system of equations:

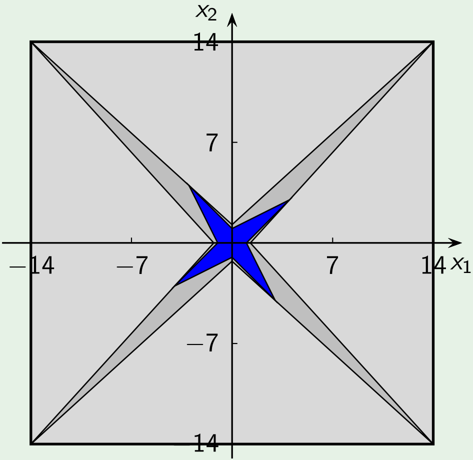
$$(R\boldsymbol{A})x = R\boldsymbol{b}.$$

Remark

- the solution set of the preconditioned systems contains Σ
- usually, we use $R \approx (A_c)^{-1}$
- then we can compute the best enclosure (Hansen, 1992, Bliek, 1992, Rohn, 1993)

Example (Barth & Nuding, 1974))

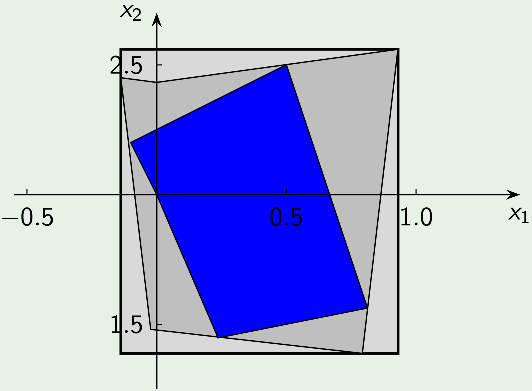
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$



Interval Linear Equations

Example (typical case)

$$\begin{pmatrix} [6,7] & [2,3] \\ [1,2] & -[4,5] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [6,8] \\ -[7,9] \end{pmatrix}$$



Interval Linear Inequalities

Interval Linear Inequalities

Let $\boldsymbol{A} \in \mathbb{IR}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{IR}^{m}$. The family of systems

$$Ax \leq b, \quad A \in \boldsymbol{A},\ b \in \boldsymbol{b}.$$

is called interval linear inequalities and abbreviated as $\boldsymbol{A}x \leq \boldsymbol{b}$.

Solution set

The solution set is defined

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \boldsymbol{A} \exists b \in \boldsymbol{b} : Ax \leq b\}.$$

Theorem (Gerlach, 1981)

A vector $x \in \mathbb{R}^n$ is a solution of $\boldsymbol{A}x \leq \boldsymbol{b}$ if and only if

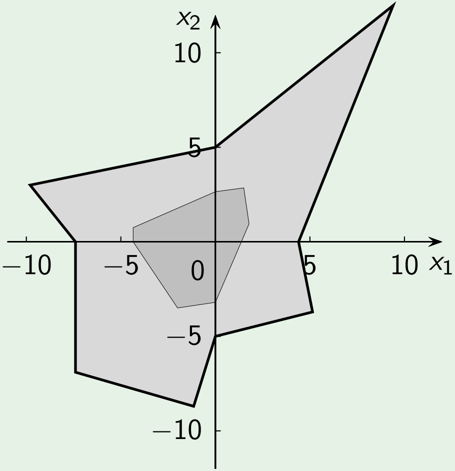
$$A_c x \leq A_\Delta |x| + \overline{b}.$$

Corollary

An $x \in \mathbb{R}^n$ is a solution of $\boldsymbol{A}x \leq \boldsymbol{b}$, $x \geq 0$ if and only if $\underline{A}x \leq \overline{b}$, $x \geq 0$.

Example of the Solution Set

Example (An interval polyhedron)



$$\begin{pmatrix} -[2,5] & -[7,11] \\ [1,13] & -[4,6] \\ [5,8] & [-2,1] \\ -[1,4] & [5,9] \\ -[5,6] & -[0,4] \end{pmatrix} x \leq \begin{pmatrix} [61,63] \\ [19,20] \\ [15,22] \\ [24,25] \\ [26,37] \end{pmatrix}$$

- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,

Strong Solution

Strong Solution

A vector $x \in \mathbb{R}^n$ is a strong solution to $\boldsymbol{A}x \leq \boldsymbol{b}$ if it solves $Ax \leq b$ for every $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

Theorem (Rohn & Kreslová, 1994)

A vector $x \in \mathbb{R}^n$ is a strong solution iff there are $x^1, x^2 \in \mathbb{R}^n$ such that

$$x = x^1 - x^2,\ \overline{A}x^1 - \underline{A}x^2 \leq \underline{b},\ x^1 \geq 0,\ x^2 \geq 0.$$

Theorem (Machost, 1970)

A vector $x \in \mathbb{R}^n$ is a strong solution $\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$ iff it solves

$$\overline{A}x \leq \underline{b},\ x \geq 0.$$

Theorem (Rohn & Kreslová, 1994)

$\boldsymbol{A}x \leq \boldsymbol{b}$ has a strong solution iff $Ax \leq b$ is solvable $\forall A \in \boldsymbol{A}, \forall b \in \boldsymbol{b}$.

No analogy for interval equations ($x + y = [1,2],\ x - y = [2,3]$).

Regularity

Definition (Regularity)

$\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is regular if each $A \in \boldsymbol{A}$ is nonsingular.

Theorem (Poljak & Rohn, 1988)

Checking regularity of an interval matrix is co-NP-hard.

Forty equivalent conditions for regularity of $\boldsymbol{A}$ by Rohn (2010), e.g.,

1. The system $|A_c x| \leq A_\Delta |x|$ has the only solution $x = 0$.
2. $\det(A_c - \operatorname{diag}(y) A_\Delta \operatorname{diag}(z))$ has constant sign for each $y, z \in \{\pm 1\}^n$.
3. $A_c x - \operatorname{diag}(y) A_\Delta |x| = y$ is solvable for each $y \in \{\pm 1\}^n$.

Theorem (Beeck, 1975)

If $\rho(|(A_c)^{-1}| A_\Delta) < 1$, *then* $\boldsymbol{A}$ *is regular.*

Necessary Condition

If $0 \in \boldsymbol{A}x$ for some $0 \neq x \in \mathbb{R}^n$, then $\boldsymbol{A}$ is not regular. (Try $x := (A_c)^{-1}_{*i}$)

Eigenvalues of Interval Matrices

Eigenvalues

- For $A \in \mathbb{R}^{n\times n}$, $A = A^T$, denote its eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$.
- Let for $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, denote its eigenvalue sets

$$\lambda_i(\boldsymbol{A}) = \{\lambda_i(A) \colon A \in \boldsymbol{A},\ A = A^T\}, \quad i = 1, \dots, n.$$

Theorem

- *Checking whether $0 \in \boldsymbol{\lambda}_i(\boldsymbol{A})$ for some $i = 1, \dots, n$ is NP-hard.*
- *We have the following enclosures for the eigenvalue sets*

$$\lambda_i(\boldsymbol{A}) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)], \quad i = 1, \dots, n.$$

- *By Hertz (1992)*

$$\overline{\lambda}_1(\boldsymbol{A}) = \max_{z\in\{\pm 1\}^n} \lambda_1(A_c + \operatorname{diag} z A_\Delta \operatorname{diag} z),$$

$$\underline{\lambda}_n(\boldsymbol{A}) = \min_{z\in\{\pm 1\}^n} \lambda_n(A_c - \operatorname{diag} z A_\Delta \operatorname{diag} z).$$

Next Section

Introduction

Linear programming – three basic forms

$$f(A, b, c) \equiv \min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$
$$f(A, b, c) \equiv \min \ c^T x \ \text{ subject to } \ Ax \leq b,$$
$$f(A, b, c) \equiv \min \ c^T x \ \text{ subject to } \ Ax \leq b, \ x \geq 0.$$

Interval linear programming

Family of linear programs with $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$, in short

$$f(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \equiv \min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x \overset{(\leq)}{=} \boldsymbol{b}, \ (x \geq 0).$$

The three forms are not transformable between each other!

Main goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

Optimal Value Range

Definition

$$\underline{f} := \min\ f(A,b,c)\ \text{ subject to }\ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},$$
$$\overline{f} := \max\ f(A,b,c)\ \text{ subject to }\ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}.$$

Observation

If $f(A,b,c)$ is continuous on $\boldsymbol{A} \times \boldsymbol{b} \times \boldsymbol{c}$, then $\underline{f}$ and $\overline{f}$ are finite and $f(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) = [\underline{f},\overline{f}]$.

Example (Bereanu, 1978)

$$\max\ x_1\ \text{ subject to }\ x_1 \le [1,2],\ [-1,1]x_1 \le 0,\ -x_1 \le 0.$$

The image of the optimal value is $\{0\} \cup [1,2]$.

Open problems

How many components of $f(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$? Always closed?

Optimal Value Range

Theorem (Wets, 1985, Mostafaee et al., 2016)

Suppose for type $(\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0)$ *that both interval linear systems*

$$\boldsymbol{A}x = 0,\ x \geq 0,\ \boldsymbol{c}^T x \leq 0$$

and

$$\boldsymbol{A}^T y \leq 0,\ \boldsymbol{b}^T y \geq 0$$

have only trivial solution. Then $f(A, b, c)$ *is continuous on* $\boldsymbol{A} \times \boldsymbol{b} \times \boldsymbol{c}$.

Theorem

It is NP-hard to check if the value f *is attained for a given* $f \in [\underline{f}, \overline{f}]$.

Optimal Value Range

Theorem (Vajda, 1961)

We have for type ($\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$)

$$\underline{f} = \min \underline{c}^T x \ \ \textit{subject to} \ \ \underline{A}x \leq \overline{b},\ x \geq 0,$$
$$\overline{f} = \min \overline{c}^T x \ \ \textit{subject to} \ \ \overline{A}x \leq \underline{b},\ x \geq 0.$$

Theorem (Machost, 1970, Rohn, 1984)

We have for type ($\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$)

$$\underline{f} = \min \underline{c}^T x \ \ \textit{subject to} \ \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0,$$
$$\overline{f} = \max_{s \in \{\pm 1\}^m} f(A_c - \mathrm{diag}(s)A_\Delta, b_c + \mathrm{diag}(s)b_\Delta, \overline{c}).$$

Theorem (Rohn (1997), Gabrel et al. (2008))

- *checking $\overline{f} = \infty$ is NP-hard*
- *checking $\overline{f} \geq 1$ is strongly NP-hard (with A, c crisp and $\overline{f} < \infty$)*

Optimal Value Range

Example (A Classification Problem)

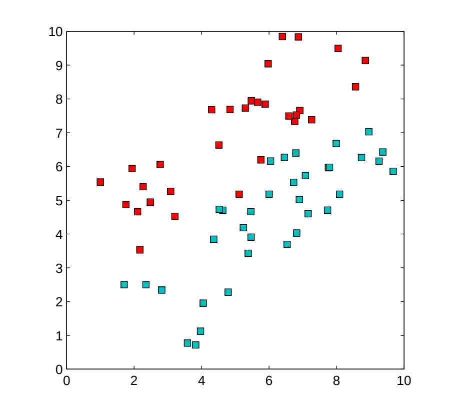
Find a separating hyperplane $a^T x = b$ for two sets of points $\{x_1, \ldots, x_m\} \subset \mathbb{R}^n$ and $\{y_1, \ldots, y_k\} \subset \mathbb{R}^n$. This can be formulated as a linear program

$$
\begin{aligned}
\min \quad & 1^T u + 1^T v \\
\text{subject to} \quad & a^T x_i - b \geq 1 - u_i, \quad i = 1, \ldots, m, \\
& a^T y_j - b \leq -(1 - v_j), \quad j = 1, \ldots, k, \\
& u, v \geq 0.
\end{aligned}
$$

- If the optimal value is zero, then the points can be separated and the optimal solution gives the separating hyperplane.
- If the optimal value is positive, then the points cannot be separated, but the optimal value approximates the minimum number of misclassified points and the optimal solution gives the corresponding hyperplane.

Example (A Classification Problem)

- For interval $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{IR}^n$ and $\mathbf{y}_1, \ldots, \mathbf{y} \in \mathbb{IR}^n$, $\underline{f}$ and $\overline{f}$ give approximately the lowest and highest number of misclassified points.
- Two sets of 30 and 35 randomly generated interval data in $\mathbb{R}^2$. We compute $\underline{f} = 0$ and $\overline{f} = 8.2$ (for the midpoint data $f = 3.15$).



Optimal Solution Set

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \;\; \text{subject to} \;\; Ax = b, \; x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, and $c \in \boldsymbol{c}$ such that

$$Ax = b, \; x \geq 0, \; A^T y \leq c, \; c^T x = b^T y,$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

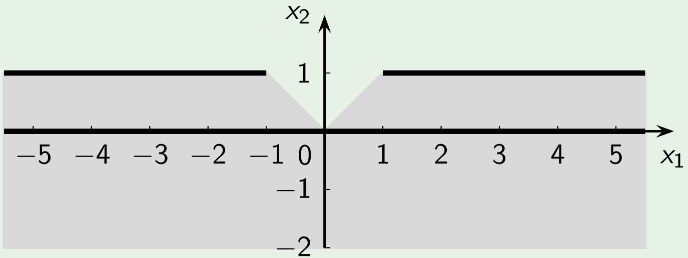
Optimal Solution Set

Example (Garajová, 2016)

The optimal solution set may be disconnected and nonconvex.

Consider the interval LP problem

$$\max\ x_2 \ \text{ subject to } \ [-1,1]x_1 + x_2 \le 0,\ x_2 \le 1.$$



Optimal Solution Set

Theorem (Garajová, H., 2016)

The set of optimal solutions $\mathcal{S}$ of the interval linear program (with real A)

$$\min\ \boldsymbol{c}^T x \ \text{ subject to }\ Ax = \boldsymbol{b},\ x \geq 0$$

is a path-connected union of at most 2^n convex polyhedra.

Observation

If b is real in addition, then $\mathcal{S}$ is formed by a union of some faces of the feasible set.

Open Problems

- More about topology of the optimal solution set $\mathcal{S}$ (Is it always polyhedral?),
- characterization of $\mathcal{S}$,
- tight approximation of $\mathcal{S}$.

Basis Stability

Definition

The interval linear programming problem

$$\min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x = \boldsymbol{b},\ x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\underline{f} = \min\ \underline{c}_B^T x \ \text{ subject to } \ \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$

$$\overline{f} = \max\ \overline{c}_B^T x \ \text{ subject to } \ \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Moreover, $f(A, b, c) = c_B^T A_B^{-1} b$ is continuous and $f(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = [\underline{f}, \overline{f}]$.

Under the unique B-stability, the set of all optimal solutions reads

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

(Otherwise each realization has at least one optimal solution in this set.)

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C1

- C1 says that $\boldsymbol{A}_B$ is regular;
- co-NP-hard problem;
- Beeck's sufficient condition: $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$.

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C2

- C2 says that the solution set to $\boldsymbol{A}_B x_B = \boldsymbol{b}$ lies in $\mathbb{R}^n_+$;
- sufficient condition: check of some enclosure to $\boldsymbol{A}_B x_B = \boldsymbol{b}$.

Basis Stability

Non-interval case

Basis B is optimal iff

- C1. A_B is non-singular;
- C2. $A_B^{-1} b \geq 0$;
- C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

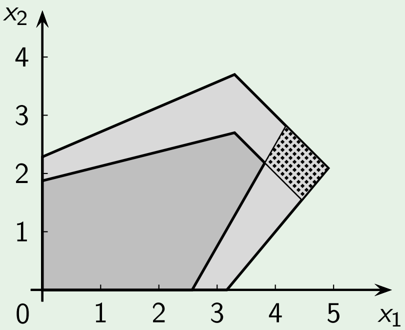
Condition C3

- C2 says that $\mathbf{A}_N^T y \leq \mathbf{c}_N, \ \mathbf{A}_B^T y = \mathbf{c}_B$ is strongly feasible;
- co-NP-hard problem;
- sufficient condition:
  $(\mathbf{A}_N^T)\mathbf{y} \leq \underline{c}_N$, where $\mathbf{y}$ is an enclosure to $\mathbf{A}_B^T y = \mathbf{c}_B$.

Example

Consider an interval linear program

$$\max\ ([5,6],[1,2])^T x \ \text{ s.t. } \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix},\ x \ge 0.$$



- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area

Basis Stability – Interval Right-Hand Side

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$ for each $b \in \boldsymbol{b}$.

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Condition C1

- C1 and C3 are trivial
- C2 is simplified to

$$\underline{A_B^{-1}\boldsymbol{b}} \geq 0,$$

  which is easily verified by interval arithmetic
- overall complexity: polynomial

Basis Stability – Interval Objective Function

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$ for each $c \in \boldsymbol{c}$

Condition C1

- C1 and C2 are trivial
- C3 is simplified to

$$A_N^T y \leq \boldsymbol{c}_N,\ A_B^T y = \boldsymbol{c}_B$$

or,

$$\overline{(A_N^T A_B^{-T})\boldsymbol{c}_B} \leq \underline{c}_N.$$

- overall complexity: polynomial

Next Section

Applications

Real-life applications

- Transportation problems with uncertain demands, suppliers, and/or costs.
- Networks flows with uncertain capacities.
- Diet problems with uncertain amounts of nutrients in foods.
- Portfolio selection with uncertain rewards.
- Matrix games with uncertain payoffs.

Technical applications

- Tool for global optimization.
- Measure of sensitivity of linear programs.

Verification

- Handle rigorously numerics of real-valued linear programs.

Applications – Diet Problem

Example (Stigler's Nutrition Model)

`http://www.gams.com/modlib/libhtml/diet.htm`.

- $n = 20$ different types of food,
- $m = 9$ nutritional demands,
- a_{ij} is the the amount of nutrient j contained in one unit of food i,
- b_i is the required minimal amount of nutrient j,
- c_j is the price per unit of food j,
- minimize the overall cost

The model reads

$$\min c^T x \ \text{ subject to } \ Ax \geq b, \ x \geq 0.$$

The entries a_{ij} are not stable!

Applications – Diet Problem

Example (Stigler's Nutrition Model)

Nutritive value of foods (per dollar spent)

	calorie (1000)	protein (g)	calcium (g)	iron (mg)	vitamin-a (1000iu)	vitamin-b1 (mg)	vitamin-b2 (mg)	niacin (mg)	vitamin-c (mg)
wheat	44.7	1411	2.0	365		55.4	33.3	441	
cornmeal	36	897	1.7	99	30.9	17.4	7.9	106	
cannedmilk	8.4	422	15.1	9	26	3	23.5	11	60
margarine	20.6	17	.6	6	55.8	.2			
cheese	7.4	448	16.4	19	28.1	.8	10.3	4	
peanut-b	15.7	661	1	48		9.6	8.1	471	
lard	41.7				.2		.5	5	
liver	2.2	333	.2	139	169.2	6.4	50.8	316	525
porkroast	4.4	249	.3	37		18.2	3.6	79	
salmon	5.8	705	6.8	45	3.5	1	4.9	209	
greenbeans	2.4	138	3.7	80	69	4.3	5.8	37	862
cabbage	2.6	125	4	36	7.2	9	4.5	26	5369
onions	5.8	166	3.8	59	16.6	4.7	5.9	21	1184
potatoes	14.3	336	1.8	118	6.7	29.4	7.1	198	2522
spinach	1.1	106		138	918.4	5.7	13.8	33	2755
sweet-pot	9.6	138	2.7	54	290.7	8.4	5.4	83	1912
peaches	8.5	87	1.7	173	86.8	1.2	4.3	55	57
prunes	12.8	99	2.5	154	85.7	3.9	4.3	65	257
limabeans	17.4	1055	3.7	459	5.1	26.9	38.2	93	
navybeans	26.9	1691	11.4	792		38.4	24.6	217	

Applications – Diet Problem

Example (Stigler's Nutrition Model)

If the entries a_{ij} are known with 10% accuracy, then

- the problem is not basis stable
- the minimal cost ranges in $[0.09878, 0.12074]$,
- the interval enclosure of the solution set is

$$[0, 0.0734],\ [0, 0.0438],\ [0, 0.0576],\ [0, 0.0283],\ [0, 0.0535],\ [0, 0.0315],\ [0, 0.0339],$$
$$[0, 0.0300],\ [0, 0.0246],\ [0, 0.0337],\ [0, 0.0358],\ [0, 0.0387],\ [0, 0.0396],\ [0, 0.0429],$$
$$[0, 0.0370],\ [0, 0.0443],\ [0, 0.0290],\ [0, 0.0330],\ [0, 0.0472],\ [0, 0.1057].$$

If the entries a_{ij} are known with 1% accuracy, then

- the problem is basis stable
- the minimal cost ranges in $[0.10758, 0.10976]$,
- the interval hull of the solution set is

$$x_1 = [0.0282, 0.0309],\ x_8 = [0.0007, 0.0031],\ x_{12} = [0.0110, 0.0114],$$
$$x_{15} = [0.0047, 0.0053],\ x_{20} = [0.0600, 0.0621].$$

Robust Interval Linear Programming

Consider the interval LP problem

$$\min \; c^T x \; \text{ subject to } \; \boldsymbol{A}x \le \boldsymbol{b}, \; x \ge 0.$$

The robust counterpart

$$\min \; c^T x \; \text{ subject to } \; Ax \le b, \; x \ge 0, \; \forall A \in \boldsymbol{A}, \; b \in \boldsymbol{b}$$

takes the form

$$\min \; c^T x \; \text{ subject to } \; \overline{A}x \le \underline{b}, \; x \ge 0.$$

Consider the interval LP problem

$$\min \; c^T x \; \text{ subject to } \; \boldsymbol{A}x \le \boldsymbol{b}.$$

The robust counterpart

$$\min \; c^T x \; \text{ subject to } \; Ax \le b, \; \forall A \in \boldsymbol{A}, \; b \in \boldsymbol{b}$$

takes the form

$$\min \; c^T x^1 - c^T x^2 \; \text{ subject to } \; \overline{A}x^1 - \underline{A}x^2 \le \underline{b}, \; x^1, x^2 \ge 0.$$

Verification – Motivation

Example (Rump, 1988)

Consider the expression

$$f = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b},$$

with

$$a = 77617, \quad b = 33096.$$

Calculations from 80s gave

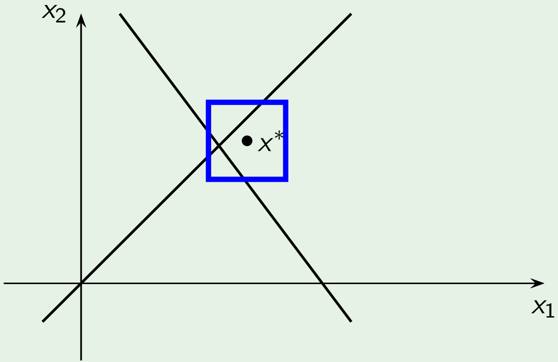
single precision	$f \approx 1.172603\ldots$
double precision	$f \approx 1.1726039400531\ldots$
extended precision	$f \approx 1.172603940053178\ldots$
the **true** value	$f = -0.827386\ldots$

Verification

Verification of a system of linear equations

Given a real system $Ax = b$ and x^* approximate solution, find $x^* \in \boldsymbol{x} \in \mathbb{IR}^n$ such that $A^{-1}b \in \boldsymbol{x}$.

Example



Verification in Linear Programming

Consider a linear program

$$\min\ c^T x \ \text{ subject to } \ Ax = b,\ x \geq 0.$$

Let B^* be an optimal basis, f^* optimal value and x^* optimal solution. All these are numerically computed.

Verification of the optimal basis (Jansson, 1988)

- confirmation that B^* is (unique) optimal basis,

Verification of the optimal value (Neumaier & Shcherbina, 2004)

- finding $f^* \in \boldsymbol{f} \in \mathbb{IR}$ such that $\boldsymbol{f}$ contains the optimal value,

Verification of the optimal solution

- finding $x^* \in \boldsymbol{x} \in \mathbb{IR}^n$ such that $\boldsymbol{x}$ contains the (unique) optimal solution.

Verification of Optimal Basis

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Verification of condition C2

- Compute verification interval $\boldsymbol{x}_B$ for $A_B x_B = b$,
- check $\underline{x}_B \geq 0$ (resp. $\underline{x}_B > 0$ for uniqueness)

Verification of condition C3

- Compute verification interval $\boldsymbol{y}$ for $A_B^T y = c_B$,
- check $c_N^T - \boldsymbol{y}^T A_N \geq 0$ (resp. $c_N^T - \boldsymbol{y}^T A_N > 0$ for uniqueness).

Conclusion

Conclusion

Interval linear programming provides techniques for

- studying effects of data variations on optimal value and optimal solutions
- processing state space of parameters
- calculating bounds
- handling numerical errors

References

M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann.
Linear Optimization Problems with Inexact Data.
Springer, New York, 2006.

M. Hladík.
Interval linear programming: A survey.
In Z. A. Mann, editor, *Linear Programming – New Frontiers in Theory and Applications*, chapter 2, pages 85–120. Nova Science Publishers, 2012.