The Worst Case Finite Optimal Value in Interval Linear Programming

Milan Hladík

Department of Applied Mathematics,
Faculty of Mathematics and Physics,
Charles University, Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

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Introduction

Linear programming – three basic forms

$$f(A,b,c) \equiv \min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

$$f(A,b,c) \equiv \min \ c^T x \ \text{ subject to } \ Ax \leq b,$$

$$f(A,b,c) \equiv \min \ c^T x \ \text{ subject to } \ Ax \leq b, \ x \geq 0.$$

Interval data (deterministic)

Interval matrix $\boldsymbol{A}$, interval vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. For example,

$$\boldsymbol{A} = \begin{pmatrix} [2,3] & [-1,1] \\ 4 & [1,2] \end{pmatrix}.$$

Interval linear programming

Family of linear programs with $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$, in short

$$f(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) \equiv \min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x \overset{(\leq)}{=} \boldsymbol{b}, \ (x \geq 0).$$

The three forms are not transformable between each other!

Optimal Value Range

Definition (Best and worst case optimal values)

$$\underline{f} := \min \ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A}, \ b \in \boldsymbol{b}, \ c \in \boldsymbol{c},$$
$$\overline{f} := \max \ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A}, \ b \in \boldsymbol{b}, \ c \in \boldsymbol{c}.$$

It may $\overline{f} = \infty$ due to infeasibility of some realization!

But sometimes we know a priori that the LP problems are feasible.

Example

Transportation problem.

Definition (Worst case finite optimal values)

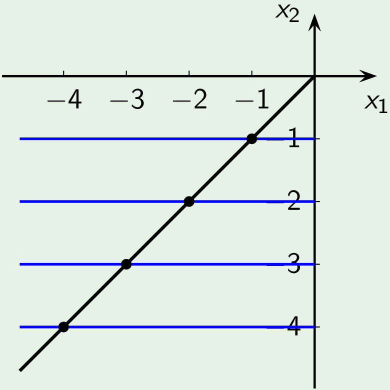
$$\overline{f}_{fin} := \max \ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A}, \ b \in \boldsymbol{b}, \ c \in \boldsymbol{c}, \ f(A,b,c) < \infty.$$

Worst Case Finite Optimal Value: General Properties

Example ($\overline{f}_{fin} = \infty$)

Consider the interval LP problem

$$\min \ -x_1 \ \text{ subject to } \ [0,1]x_2 = -1, \ x_1 - x_2 = 0, \ x_1, x_2 \le 0.$$



By direct inspection, $\boldsymbol{f} = [1, \infty)$ and $\overline{f}_{fin} = \infty$.

Worst Case Finite Optimal Value: General Properties

Proposition

Let $g(A,b,c)$ be the dual optimal value. Then

$$\overline{f}_{fin} = \max\ g(A,b,c) \ \ \textit{subject to} \ \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},\ g(A,b,c) < \infty.$$

Advantage

"max min" optimization problem is reduced to "max max" problem

$$\overline{f}_{fin} = \max\ b^T y \ \ \text{subject to} \ \ y \in N(A^T, c),\ M(A,b) \neq \emptyset,$$
$$A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}.$$

Two Special Cases

Proposition (Interval objective function)

If A and b are real, then computation of $\overline{f}_{fin}$ is a polynomial problem.

- For constrains $Ax = b,\ x \geq 0$:

$$\overline{f}_{fin} = \max\ b^T y \ \text{ subject to }\ Ax = b,\ x \geq 0,\ A^T y \leq \overline{c},$$

- For constrains $Ax \leq b$:

$$\overline{f}_{fin} = \max\ b^T y \ \text{ subject to }\ Ax \leq b,\ \underline{c} \leq A^T y \leq \overline{c},\ y \leq 0.$$

- For constrains $Ax \leq b,\ x \geq 0$:

$$\overline{f}_{fin} = \max\ b^T y \ \text{ subject to }\ Ax \leq b,\ x \geq 0,\ A^T y \leq \overline{c},\ y \leq 0.$$

Proposition

If A and c are real, then checking $\overline{f}_{fin} > 0$ is NP-hard (for each type).

Basis Approach

Consider the LP problem

$$f(A, b, c) = \min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0.$$

A basis B is optimal if and only if

$$A_B^{-1} b \geq 0, \tag{1a}$$

$$c_N^T - c_B^T A_B^{-1} A_N \geq 0^T. \tag{1b}$$

Worst optimal value achievable at B

$$\max \ c_B^T A_B^{-1} b \ \text{ subject to } \ (1), \ A \in \boldsymbol{A}, \ b \in \boldsymbol{b}, \ c \in \boldsymbol{c}.$$

Basis Approach with Real A

Worst optimal value achievable at B

$$\max \ c_B^T A_B^{-1} b \ \text{ subject to } \ A_B^{-1} b \geq 0, \tag{2a}$$
$$c_N^T - c_B^T A_B^{-1} A_N \geq 0^T \tag{2b}$$
$$b \in \boldsymbol{b}, \ c \in \boldsymbol{c}. \tag{2c}$$

Corollary

If A is real, then $\overline{f}_{fin} < \infty$.

Proposition

If A is real, then solving (2) is NP-hard.

Proposition

If A, b are real or A, c are real, then solving (2) is polynomial.

Basis Approach with Real A and c

Basis-by-basis inspection

1. Inspect all possibly optimal bases one by one.
2. Start with one possibly optimal basis and repeatedly inspect the neighboring bases.
   (The graph of possibly optimal bases is connected.)
3. The set $\boldsymbol{b}$ decomposes into convex polyhedral regions related to possibly optimal bases.

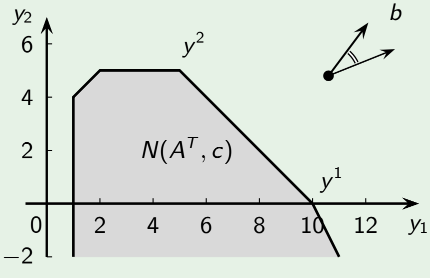
Basis Approach with Real A and c – Example

Example

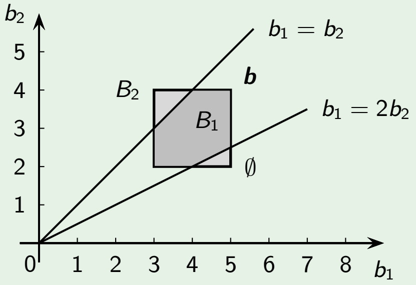
Consider the LP problem with data

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [3,5] \\ [2,4] \end{pmatrix}, \quad c = \begin{pmatrix} 10 & 20 & 5 & 3 & 1 \end{pmatrix}^T.$$

Two possibly optimal bases, $B_1 = \{1,2\}$ and $B_2 = \{1,3\}$.



Dual LP problem.



$\boldsymbol{b}$ consists of basis stable regions.

References

R. Cerulli, C. D'Ambrosio, and M. Gentili.
Best and worst values of the optimal cost of the interval transportation problem.
In A. Sforza and C. Sterle, editors, *Optimization and Decision Science: Methodologies and Applications*, volume 217 of *Springer Proceedings in Mathematics & Statistics*, pages 367–374. Springer, Cham, 2017.

M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann.
Linear Optimization Problems with Inexact Data.
Springer, New York, 2006.

M. Hladík.
Interval linear programming: A survey.
In Z. A. Mann, editor, *Linear Programming – New Frontiers in Theory and Applications*, chapter 2, pages 85–120. Nova Science Publishers, 2012.