P-Completeness of Testing Solutions of Parametric Interval Linear Systems

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P-Completeness

P-complete problems

- Informally: the hardest problems in P (polynomial); difficult to parallelize effectively
- More formally:
  - NC is the set of problems that can be solved in polylogarithmic time with a polynomial number of processors
  - P-complete problems problems in P and every problem in P can be reduced to it by an NC-reduction
  - It is believed that $NC \neq P$ and the P-complete problems lie outside NC

Examples of P-complete problems

- Linear programming (even max flow, zero-sum games)
- Solvability of linear inequalities
- ...

Notation

Interval matrix

An interval matrix

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A^c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $m \times n$ interval matrices: $\mathbb{IR}^{m \times n}$.

A symmetric interval matrix

$$\boldsymbol{A}^S := \{A \in \boldsymbol{A} : A = A^T\}.$$

Without loss of generality assume that $\underline{A} = \underline{A}^T$, $\overline{A} = \overline{A}^T$, and $\boldsymbol{A}^S \neq \emptyset$.

Notation

Parametric interval system

Consider a parametric interval linear system

$$A(p)x = b(p),$$

in which parameters have a linear structure

$$A(p) = \sum_{k=1}^{K} A^{(k)} p_k, \quad b(p) = \sum_{k=1}^{K} b^{(k)} p_k.$$

Herein,

- $A^{(1)}, \dots, A^{(K)} \in \mathbb{R}^{n \times n}$ and $b^{(1)}, \dots, b^{(K)} \in \mathbb{R}^n$ are fixed,
- parameters $p_1, \dots, p_K$ come from interval domains $\boldsymbol{p}_1, \dots, \boldsymbol{p}_K \in \mathbb{IR}$.

Remark

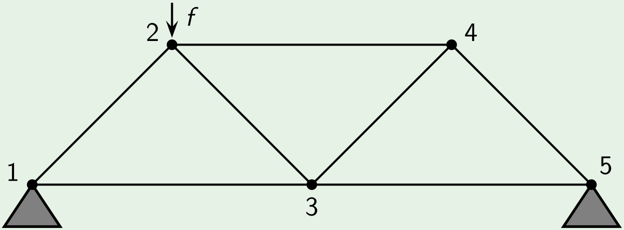
A symmetric interval matrix is a special linear parametric matrix

$$A(\boldsymbol{p}) = \boldsymbol{p}_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{p}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \boldsymbol{p}_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Parametric Interval Linear Systems – Example

Example (Displacements of a truss structure (Skalna, 2006))

The 7-bar truss structure subject to downward force.
The stiffnesses s_{ij} of bars are uncertain.
The displacements d of the nodes, are solutions of the system $Kd = f$, where f is the vector of forces.



Parametric Linear Interval Systems – Example

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$$K = \begin{pmatrix} \frac{s_{12}}{2} + s_{13} & -\frac{s_{12}}{2} & -\frac{s_{12}}{2} & -s_{13} & 0 & 0 & 0 \\ -\frac{s_{21}}{2} & \frac{s_{21}+s_{23}}{2} + s_{24} & \frac{s_{21}-s_{23}}{2} & -\frac{s_{23}}{2} & \frac{s_{23}}{2} & -s_{24} & 0 \\ -\frac{s_{21}}{2} & \frac{s_{21}-s_{23}}{2} & \frac{s_{21}+s_{23}}{2} & \frac{s_{23}}{2} & -\frac{s_{23}}{2} & 0 & 0 \\ -s_{31} & -\frac{s_{32}}{2} & \frac{s_{32}}{2} & s_{31} + \frac{s_{32}+s_{34}}{2} + s_{35} & \frac{s_{34}-s_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\ 0 & \frac{s_{32}}{2} & -\frac{s_{32}}{2} & \frac{s_{34}-s_{32}}{2} & \frac{s_{34}+s_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\ 0 & -s_{42} & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & s_{42} + \frac{s_{43}+s_{45}}{2} & 0 \\ 0 & 0 & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & 0 & \frac{s_{43}+s_{45}}{2} \end{pmatrix}$$

Problem Formulation

Standard, symmetric and parametric solution set

Standard solution set

$$\Sigma = \{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A}\, \exists b \in \boldsymbol{b} : Ax = b\}.$$

Parametric solution set

$$\Sigma_{par} = \{x \in \mathbb{R}^n;\ \exists p \in \boldsymbol{p} : A(p)x = b(p)\}.$$

Symmetric solution set

$$\Sigma_{sym} = \{x \in \mathbb{R}^n;\ \exists p \in \boldsymbol{p}\, \exists A \in \boldsymbol{A} : Ax = b(p),\ A = A^T\}.$$

Symmetric solution set with no dependencies in the right-hand side

$$\Sigma^*_{sym} = \{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A}\, \exists b \in \boldsymbol{b} : Ax = b,\ A = A^T\}.$$

Problem formulation

Given $x^* \in \mathbb{R}^n$, how hard it is to check the following?

$$x^* \in \Sigma, \quad x^* \in \Sigma_{par}, \quad x^* \in \Sigma_{sym}, \quad x^* \in \Sigma^*_{sym}.$$

Theorem (Oettli–Prager, 1964)

A point $x \in \mathbb{R}^n$ solves $\boldsymbol{A}x = \boldsymbol{b}$ (i.e., $x \in \Sigma$) iff

$$|A^c x - b^c| \leq A^\Delta |x| + b^\Delta.$$

Corollary

Deciding $x^* \in \Sigma$ is a strongly polynomial problem and in NC.
(i.e., polynomial in the dimension, not in the input size)

Parametric Interval Systems with Linear Dependencies

Characterization of $A(p)x = b(p),\ p \in \boldsymbol{p}$

- No simple description is known.
- Description of Σ_{par} by possibly double exponential number of nonlinear inequalities by using Fourier–Motzkin elimination (Alefeld, Kreinovich, Mayer, 2003; Popova, 2015;...).
- "Infinite" characterization by Hladík (2012)

$$y^T\left(A(p^c)x - b(p^c)\right) \leq \sum_{k=1}^{K} p_k^{\Delta}\left|y^T(A^{(k)}x - b^{(k)})\right|, \quad \forall y \in \mathbb{R}^n.$$

Theorem (Popova (2009))

If $x \in \Sigma_{par}$, then it solves

$$|A(p^c)x - b(p^c)| \leq \sum_{k=1}^{K} p_k^{\Delta}|A^{(k)}x - b^{(k)}|.$$

The converse holds if no interval parameter is in more than one equation.

Parametric Interval Systems with Linear Dependencies

Theorem

Checking $x^* \in \Sigma_{par}$ is a P-complete problem.

Proof.

By reduction from checking solvability of a linear system $Ax = b$, $x \geq 0$, which is P-complete.

The condition $x^* \in \Sigma_{par}$ is equivalent to solvability

$$\sum_{k=1}^{K} (A^{(k)} p_k) x^* = \sum_{k=1}^{K} b^{(k)} p_k, \quad p_k \in \boldsymbol{p}_k$$

in variables $p_1, \ldots, p_K$. $\square$

Symmetric Solution Set

Recall the definition

$$\Sigma_{sym} = \{x \in \mathbb{R}^n;\ \exists p \in \boldsymbol{p}\, \exists A \in \boldsymbol{A} : Ax = b(p),\ A = A^T\}.$$

Theorem

Checking $x^* \in \Sigma_{sym}$ is a P-complete problem, even on a subclass of problems where the constraint matrix is tridiagonal with zero diagonal, and the right-hand side vector has at most one parameter in each entry.

Proof.

Again by reduction from checking solvability of a linear system $Ax = b$, $x \geq 0$. □

Simple Symmetric Solution Set

Symmetric solution set with no dependencies in the right-hand side

$$\Sigma^*_{sym} = \{x \in \mathbb{R}^n;\ \exists A \in \boldsymbol{A}\, \exists b \in \boldsymbol{b} : Ax = b,\ A = A^T\},$$

About

- 1986: Neumaier's letter to Rohn
- 1990's: Alefeld, Kreinovich, Mayer:
  Fourier–Motzkin elimination applied in each orthant.
- 2008: Hladík: explicit description ($r \equiv -A^c x + b^c$)

$$A^\Delta |x| + b^\Delta \geq |r|,$$

$$\sum_{i,j=1}^{n} a^\Delta_{ij} |x_i x_j (p_i - q_j)| + \sum_{i=1}^{n} b^\Delta_i |x_i (p_i + q_i)| \geq \left| \sum_{i=1}^{n} r_i x_i (p_i - q_i) \right|$$

$$\forall p, q \in \{0,1\}^n \setminus \{0^n, 1^n\}, p \prec_{\mathrm{lex}} q$$

- 2012, 2015: Mayer: further simplification

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Open problem

Is the problem of checking $x^* \in \Sigma^*_{sym}$ P-complete?

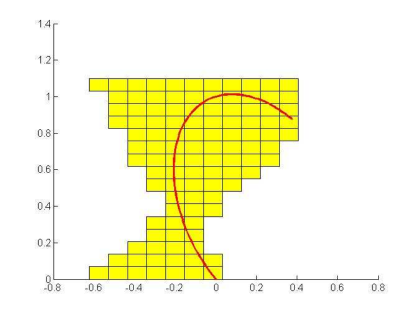
However, the problem of finding the "best" certificate is P-complete.

Theorem

The following problem is P-complete: Given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{IR}^n$, symmetric $F \in \mathbb{R}^{n\times n}$, and $x^* \in \mathbb{R}^n$, among symmetric matrices $A \in \boldsymbol{A}$ for which $Ax^* \in \boldsymbol{b}$, find the matrix for which $\operatorname{tr}(AF)$ is the largest possible.

Consequences and Conclusion

- Testing $x^* \in \Sigma_{par}$ or $x^* \in \Sigma_{sym}$ is a polynomial problem, but cannot be done "very" fast and is hard to parallelize.
- The same for checking $\boldsymbol{x} \subseteq \Sigma_{par}$ or $\boldsymbol{x} \cap \Sigma_{par} = \emptyset$.
- But this is a typical task in CSP when solved by Branch & Prune.



- Therefore nor the basic tasks (pruning boxes etc.) in CSP can be done "very effectively".

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