Recognizing Pseudoconvexity of a Function on an Interval Domain

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Pseudoconvexity

Motivation

Convexity has many nice properties in the context of optimization.
What about its generalizations?

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and $\mathcal{S} \subset \mathbb{R}^n$ an open convex set.
Then $f(x)$ is *pseudoconvex* on $\mathcal{S}$ if for every $x, y \in \mathcal{S}$ we have

$$\nabla f(x)^T (y - x) \geq 0 \quad \Rightarrow \quad f(y) \geq f(x).$$

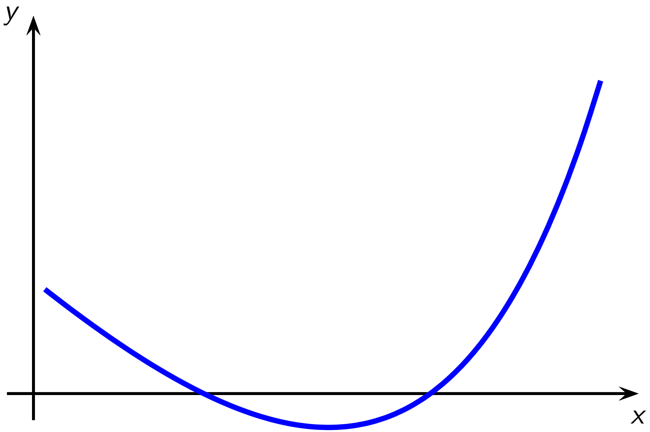
Key Properties

Minimizing pseudoconvex objective functions on convex feasible sets,

- each stationary point is a global minimum,
- each local minimum is a global minimum,
- the optimal solution set is convex.

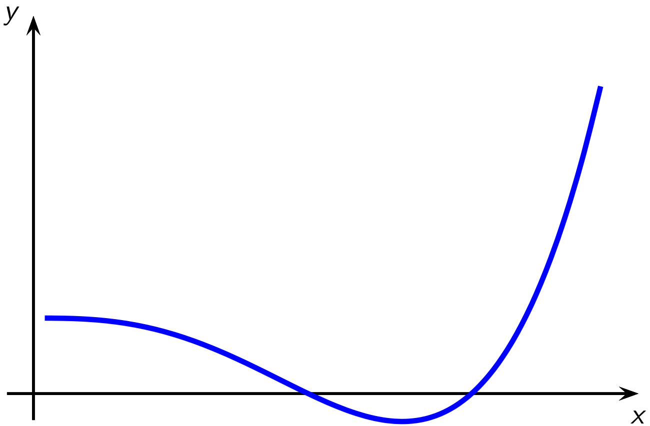
Illustration

Convex function



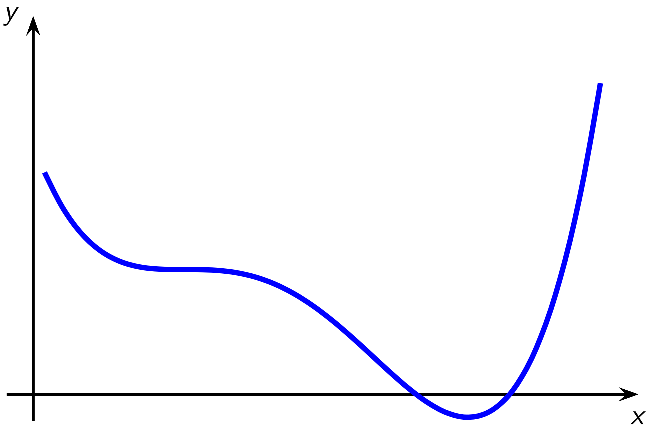
Illustration

Pseudoconvex function



Illustration

Quasiconvex function



Pseudoconvexity

Problem Formulation

Given a box $\boldsymbol{x} = [\underline{x}, \overline{x}]$ in $\mathbb{R}^n$ and differentiable $f : \mathbb{R}^n \to \mathbb{R}$.

The question: Is $f(x)$ pseudoconvex on $\boldsymbol{x}$?

Why testing pseudoconvexity on a box?

In global optimization, feasible sub-domains have often the form of boxes. Verifying pseudoconvexity can help to process a given box (for example, by local search).

Theorem (Ahmadi et al., 2013)

Deciding pseudoconvexity is NP-hard on a class of quartic polynomials.

Pseudoconvexity Characterizations

Theorem (Mereau and Paquet, 1974)

The function $f(x)$ is pseudoconvex on $\boldsymbol{x}$ if there is $\alpha \geq 0$ such that

$$M_\alpha(x) := \nabla^2 f(x) + \alpha \nabla f(x) \nabla f(x)^T$$

is positive semidefinite for all $x \in \boldsymbol{x}$.

Denote

$$D(x) := \begin{pmatrix} 0 & \nabla f(x)^T \\ \nabla f(x) & \nabla^2 f(x) \end{pmatrix},$$

and by $D(x)_r$ we denote the principal leading submatrix of size r.

Theorem (Ferland, 1972)

The function $f(x)$ is pseudoconvex on $\boldsymbol{x}$ if $\det(D(x)_r) < 0$ for every $r = 2, \ldots, n+1$ and for all $x \in \boldsymbol{x}$.

Pseudoconvexity Characterizations

Theorem (Crouzeix and Ferland, 1982)

The function $f(x)$ is pseudoconvex on $\mathbf{x}$ if for each $x \in \mathbf{x}$ either $\nabla^2 f(x)$ is positive semidefinite, or $\nabla^2 f(x)$ has one simple negative eigenvalue and there is $b \in \mathbb{R}^n$ such that $\nabla^2 f(x) b = \nabla f(x)$ and $\nabla f(x)^T b < 0$.

Theorem (Crouzeix, 1998)

The function $f(x)$ is pseudoconvex on $\mathbf{x}$ if for each $x \in \mathbf{x}$ the matrix $D(x)$ is nonsingular and has exactly one simple negative eigenvalue.

Theorem (Crouzeix, 1998)

The function $f(x)$ is pseudoconvex on $\mathbf{x}$ if for each $x \in \mathbf{x}$ and every $y \neq 0$ such that $\nabla f(x)^T y = 0$ we have $y^T \nabla^2 f(x) y > 0$.

Interval Methods for Testing Pseudoconvexity

Interval Enclosures

Let $\boldsymbol{H} \in \mathbb{IR}^{n\times n}$ (interval matrix) and $\boldsymbol{g} \in \mathbb{IR}^{n}$ (interval vector) such that

$$\nabla^2 f(x) \in \boldsymbol{H} \quad \forall x \in \boldsymbol{x},$$
$$\nabla f(x) \in \boldsymbol{g} \quad \forall x \in \boldsymbol{x}.$$

- Such interval enclosures of the Hessian matrix and the gradient can be computed, e.g., by interval arithmetic using automatic differentiation.
- If every $H \in \boldsymbol{H}$ is positive semidefinite, then $f(x)$ is convex and we are done. Therefore, we focus on problems such that not every $H \in \boldsymbol{H}$ is positive semidefinite.

We will use the symmetric interval matrix

$$\boldsymbol{D} := \begin{pmatrix} 0 & \boldsymbol{g}^T \\ \boldsymbol{g} & \boldsymbol{H} \end{pmatrix}.$$

Methods Based on Mereau and Paquet

Mereau and Paquet suggest to verify positive semidefiniteness of matrices

$$M_\alpha(H,g) := H + \alpha g g^T, \quad H \in \boldsymbol{H},\ g \in \boldsymbol{g}$$

for a suitable $\alpha \geq 0$.

Direct Evaluation (MP1)

By interval arithmetic and for a suitable $\alpha \geq 0$ evaluate

$$\boldsymbol{M}(\alpha) := \boldsymbol{H} + \alpha \boldsymbol{g}\boldsymbol{g}^T.$$

Then check whether $\boldsymbol{M}(\alpha)$ is positive semidefinite.

Problems:

- Choice of α.
- Checking positive semidefiniteness of interval matrices is co-NP-hard.
- This approach does not utilize the structure of $M_\alpha(x)$.

Sufficient condition is: $\lambda_n(M(\alpha)_c) \geq \rho(M(\alpha)_\Delta)$.

Methods Based on Mereau and Paquet

Theorem

We have that $M_\alpha(H,g)$ is positive semidefinite for all $H \in \boldsymbol{H}$ and $g \in \boldsymbol{g}$ if

$$x^T(H_c + \alpha g_c g_c^T)x - |x|^T H_\Delta |x| - 2\alpha |g_c^T x| g_\Delta^T |x| \geq 0, \quad \forall x \in \mathbb{R}^n$$

Theorem

We have that $M_\alpha(H,g)$ is positive semidefinite for all $H \in \boldsymbol{H}$ and $g \in \boldsymbol{g}$ if

$$H_c - \operatorname{diag}(z) H_\Delta \operatorname{diag}(z) + \alpha(g_c g_c^T - g_c g_\Delta^T \operatorname{diag}(z) - \operatorname{diag}(z) g_\Delta g_c^T)$$

is positive semidefinite for every $z \in \{\pm 1\}^n$.

Structure-Oriented Method (MP2)

Based on the above exponential formula.

Method Based on Ferland

Ferland suggests to check that for each symmetric $D \in \boldsymbol{D}$ and for each $r = 2, \ldots, n+1$ we have $\det(D_r) < 0$.

Theorem

It is co-NP-hard to check whether $\det(D) < 0$ *for every symmetric* $D \in \boldsymbol{D}$.

The Method (F)

Check

$$\det((D_r)_c) < 0 \quad \text{and} \quad \rho(|(D_r)_c^{-1}|(D_r)_\Delta) < 1$$

for each $r = 2, \ldots, n+1$.

Method Based on Crouzeix and Ferland

For H symmetric, the condition that there is b such that $Hb = g$, $g^T b < 0$ is equivalent to

$$\det(D) = \det \begin{pmatrix} 0 & g^T \\ g & H \end{pmatrix} < 0.$$

This gives us an equivalent condition:

Theorem

The function $f(x)$ is pseudoconvex on $\boldsymbol{x}$ if for each symmetric $D \in \boldsymbol{D}$ we have $\det(D) < 0$, and each symmetric $H \in \boldsymbol{H}$ is nonsingular and has at most one simple negative eigenvalue.

The Method (CF)

The function $f(x)$ is pseudoconvex on $\boldsymbol{x}$ if

$$\det(D_c) < 0, \quad \rho(|D_c^{-1}|D_\Delta) < 1, \quad \text{and} \quad 0 < \lambda_{n-1}(H_c) - \rho(H_\Delta).$$

Method Based on Crouzeix

Ferland suggests to check that the nth largest eigenvalue of every symmetric matrix $D \in \boldsymbol{D}$ is positive.

Theorem

Checking that the nth largest eigenvalue of every symmetric matrix $D \in \boldsymbol{D}$ *is positive is a co-NP-hard problem even on the class of problems with* $\boldsymbol{g} = 0$, H_c *symmetric positive definite and entrywise nonnegative, and* H_Δ *consisting of ones.*

The Method (C)

The function $f(x)$ is pseudoconvex on $\boldsymbol{x}$ if $0 \notin \boldsymbol{g}$ and $\lambda_n(D_c) > \rho(D_\Delta)$.

Example (Random choices of $\boldsymbol{H}$ and $\boldsymbol{g}$)

n =dimension, d =radius of $\boldsymbol{H}$ and $\boldsymbol{g}$,
$\boldsymbol{H} := \boldsymbol{H} - \gamma I_n$ minimally to fail positive semidefiniteness.

		success rate (in %)					time (in 10^{-3}sec.)				
n	d	MP1	MP2	F	CF	C	MP1	MP2	F	CF	C
5	1	0	21.2	35.7	40.7	43.5	1.12	9.32	2.14	0.835	0.644
10	1	0	3.2	9.4	11.0	29.3	0.889	49.8	3.71	0.831	0.669
15	1	0	0.3	1.0	1.3	20.3	0.958	427	5.34	0.860	0.694
20	1	0	0	0	0	11.8	1.32	3085	7.43	1.20	0.775
5	0.1	47	52	66.8	67.7	65.4	0.978	6.45	2.24	0.814	0.629
10	0.1	37	50.3	61	62	56.1	3.88	193	4.38	0.936	0.662
15	0.1	26.7	45.7	54.6	55.5	41.8	109	5814	6.61	0.973	0.681
20	0.1	25	51	57	57	41	6689	280048	11.1	1.25	0.793

The winners: Crouzeix and Ferland (CF) and Crouzeix (C)

Open problems: choice of α in (MP1–2), improve (CF) and (C)

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