The Shape of the Optimal Value of a Fuzzy Linear Programming Problem

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Interval LP

Linear Programming

$$\min \; c^T x \;\text{ subject to }\; Ax = b,\; x \geq 0$$

Arising in many practical problems:

- transportation, networks, production, scheduling & planning, assignment, investment, regression, classification, approximation, zero-sum game, ...

Data often uncertain!

- imprecise measurements, estimation, discretization, ...

Interval Linear Programming Problem

A family of linear programs

$$f(A, b, c) \equiv \min \; c^T x \;\text{ subject to }\; Ax = b,\; x \geq 0,$$

where $c \in \boldsymbol{c} = [\underline{c}, \overline{c}]$, $b \in \boldsymbol{b} = [\underline{b}, \overline{b}]$, and $A \in \boldsymbol{A} = [\underline{A}, \overline{A}]$.

Assume $f(A, b, c) \in \mathbb{R}$ for all $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and , $c \in \boldsymbol{c}$.

Interval LP: Optimal Value Range

Optimal Value Range

The range of optimal values $\boldsymbol{f} = [\underline{f}, \overline{f}]$, where

$$\underline{f} \equiv \min\ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},$$

$$\overline{f} \equiv \max\ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}.$$

Theorem (Beeck (1978), Machost (1970), Rohn (1976, 1984))

We have

$$\underline{f} = \min \underline{c}^T x \ \text{ subject to } \ \underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,$$

$$\overline{f} = \sup_{s \in \{\pm 1\}^m} f(A_c - \operatorname{diag}(s)A_\Delta, b_c + \operatorname{diag}(s)b_\Delta, \overline{c}),$$

where $A_c := \frac{1}{2}(\underline{A} + \overline{A})$, $A_\Delta := \frac{1}{2}(\overline{A} - \underline{A})$ *and m is the number of equations.*

Theorem (Rohn (1997), Gabrel et al. (2008))

- *checking* $\overline{f} = \infty$ *is NP-hard*
- *checking* $\overline{f} \ge 1$ *is strongly NP-hard (with A, c crisp and $\overline{f} < \infty$)*

Interval LP: Basis Stability

Definition

The interval linear programming problem

$$\min \boldsymbol{c}^T x \quad \text{subject to} \quad \boldsymbol{A}x = \boldsymbol{b},\ x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\underline{f} = \min\ \underline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$
$$\overline{f} = \max\ \overline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Theorem

- *checking B-stability for a given basis B is co-NP-hard (Hladík, 2014), but there are sufficient conditions*
- *it is polynomial for crisp A*

Fuzzy LP

Fuzzy Linear Program

$$\widetilde{f} := \min \ \widetilde{c}^T x \ \text{ subject to } \ \widetilde{A}x = \widetilde{b},\ x \geq 0$$

α-cut is an interval linear program

$$\boldsymbol{f}_\alpha := \min \ \boldsymbol{c}_\alpha^T x \ \text{ subject to } \ \boldsymbol{A}_\alpha x = \boldsymbol{b}_\alpha,\ x \geq 0$$

Fuzzy Optimal Value

Fuzzy optimal value $\widetilde{f}$ is defined via α-cut $\boldsymbol{f}_\alpha = [\underline{\boldsymbol{f}}_\alpha, \overline{\boldsymbol{f}}_\alpha]$, where

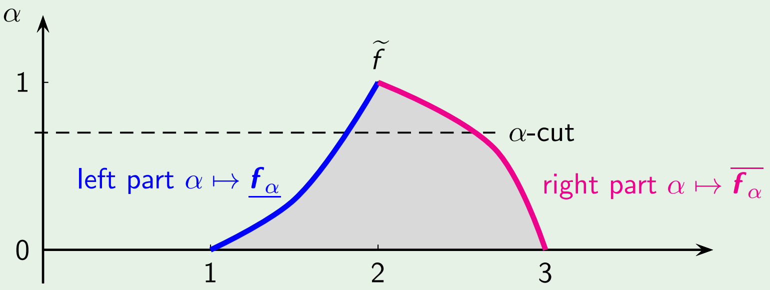
$$\underline{\boldsymbol{f}}_\alpha := \min \ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A}_\alpha,\ b \in \boldsymbol{b}_\alpha,\ c \in \boldsymbol{c}_\alpha,$$
$$\overline{\boldsymbol{f}}_\alpha := \max \ f(A,b,c) \ \text{ subject to } \ A \in \boldsymbol{A}_\alpha,\ b \in \boldsymbol{b}_\alpha,\ c \in \boldsymbol{c}_\alpha.$$

Shape of the Fuzzy Optimal Value

By the shape of $\widetilde{f}$ we mean the shape of the function $\alpha \mapsto \boldsymbol{f}_\alpha = [\underline{\boldsymbol{f}}_\alpha, \overline{\boldsymbol{f}}_\alpha]$. In particular, $\alpha \mapsto \underline{\boldsymbol{f}}_\alpha$ is the left part and $\alpha \mapsto \overline{\boldsymbol{f}}_\alpha$ the right part.

Fuzzy LP: Shape of the Optimal Value

Example



Fuzzy LP: Optimal Value

Proposition

The optimal value $\widetilde{f}$ is a well-defined fuzzy number.

Proposition

If the shape of the input coefficients in $\widetilde{A}, \widetilde{b}, \widetilde{c}$ is polynomial, then the shape of $\widetilde{f}$ is determined by a piecewisely rational polynomial function.

Proof.

On a basis stable neighbourhood we have $f(A, b, c) = c_B^T A_B^{-1} b$. $\square$

Proposition

If the optimal value $f(A, b, c)$ is continuous on $(\alpha = 0)$-cut, then the piecewise polynomial segments are continuously connected.
Otherwise, there may be jumps.

Fuzzy LP: Shape of the Optimal Value

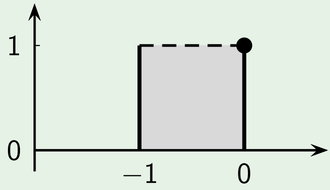
Example

Consider the fuzzy LP problem with one triangular fuzzy coefficient

$$\min \; x \;\; \text{subject to} \;\; x \geq -1, \; x \leq 0, \; [-1, 0, 1]x \geq 0.$$

- the $(\alpha = 1)$-cut of the optimal value is $\boldsymbol{f}_{\alpha=1} = -1$,
- for every $\alpha \in [0, 1)$ the α-cut reads $\boldsymbol{f}_{\alpha} = [-1, 0]$.

So, $\widetilde{f}$ is still an ordinary fuzzy number, but with an unusual shape.



Fuzzy LP: Shape of the Optimal Value

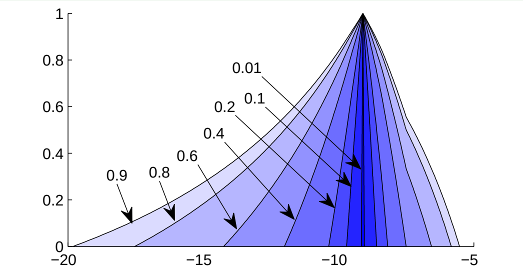
Example

$\min_{x\in\mathbb{R}^6} c^T x$ s.t. $\widetilde{A}x = b,\ x \geq 0$, where $\widetilde{A}$ has fuzzy triangular entries

$$\widetilde{A} = \begin{pmatrix} [1-\Delta, 1, 1+\Delta] & [2-\Delta, 2, 2+\Delta] & 1 & 0 & 0 & 0 \\ [1-\Delta, 1, 1+\Delta] & [1-\Delta, 1, 1+\Delta] & 0 & 1 & 0 & 0 \\ [2-\Delta, 2, 2+\Delta] & [1-\Delta, 1, 1+\Delta] & 0 & 0 & 1 & 0 \\ [3-\Delta, 3, 3+\Delta] & [1-\Delta, 1, 1+\Delta] & 0 & 0 & 0 & 1 \end{pmatrix},$$

and Δ is a parameter. The crisp-valued coefficients are

$$c = (-0.8, -1.5, 0, 0, 0, 0)^T, \quad b = (12, 7, 10, 12)^T.$$



Fuzzy LP: Polynomial Shape of the Optimal Value

Proposition

Suppose that the interval LP problem is basis stable for $\alpha = 0$. *Suppose that* $\widetilde{A}, \widetilde{b}$ *are crisp and the shape of* $\widetilde{c}$ *is described by a polynomial of degree* d.
Then the shape of $\widetilde{f}$ *is determined by a polynomial of degree* d.

Remark

The result holds analogously for the case with $\widetilde{A}, \widetilde{c}$ crisp and $\widetilde{b}$ fuzzy.

Corollary

Under assumptions of Proposition above, if $\widetilde{c}$ *has a triangular shape, then* $\widetilde{f}$ *has a triangular shape. Moreover, if* $\widetilde{c}$ *has a symmetric triangular shape, then* $\widetilde{f}$ *has a symmetric triangular shape.*

Proposition

If $\widetilde{A}, \widetilde{b}$ *are crisp and* $\widetilde{c}$ *is fuzzy triangular, then* $\widetilde{f}$ *is concave piecewise linear.*

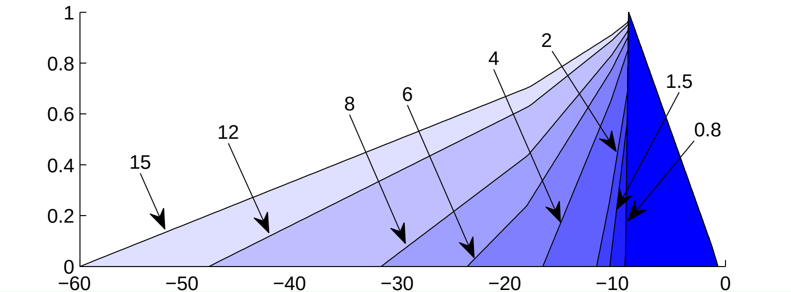
Fuzzy LP: Linear Shape of the Optimal Value

Example

A, b are crisp and $\widetilde{c}$ fuzzy triangular depending on parameter Δ

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} b &= (12, 7, 10, 12)^T, \\ \widetilde{c} &= ([-\Delta, -0.2, -0.1], \\ &\qquad [-1.55, -1.5, -0.1], 0, 0, 0, 0)^T. \end{aligned}$$

Membership function of $\widetilde{f_\Delta}$ is piecewise linear and concave in α.



Conclusion

Summary

- linear programming problems have often uncertain data in practice,
- shape of the optimal value in fuzzy linear programming (polynomial, linear, concave, ...)
- info for a decision maker what is the effect on optimal value.

References

M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann.
Linear optimization problems with inexact data.
Springer, New York, 2006.

T. Gal.
Postoptimal Analyses, Parametric Programming, and Related Topics.
McGraw-Hill, New York, 1979.

M. Hladík.
Interval linear programming: A survey.
In Z. A. Mann, editor, *Linear Programming - New Frontiers in Theory and Applications*, chap. 2, pp. 85–120. Nova Science Publ., NY, 2012.

M. Hladík.
How to determine basis stability in interval linear programming.
Optim. Lett., 8(1):375–389, 2014.

A. Mostafaee, M. Hladík, and M. Černý.
Inverse linear programming with interval coefficients.
J. Comput. Appl. Math., 292:591–608, 2016.