

AE regularity of interval matrices

Milan Hladík

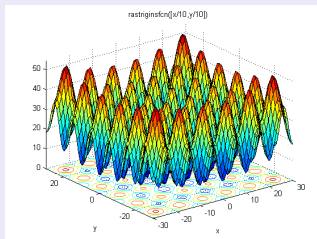
Department of Applied Mathematics
Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
<http://kam.mff.cuni.cz/~hladik/>

MatTriad 2017
Będlewo, Poland
September 25–29, 2017

Interval Matrices – Motivation

Where interval data do appear

- numerical analysis (handling rounding errors)
 - $\frac{1}{3} \in [0.333333333333333, 0.333333333333334]$
 - $\pi \in [3.1415926535897932384, 3.1415926535897932385]$.
- constraint solving and global optimization
 - find robot singularities, where it may breakdown
check joint angles $[0, 180]^\circ$
 - find minimum of $f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))$



- modelling uncertainty
 - temperature measured in Będlewo, $20^\circ\text{C} \pm 1^\circ\text{C}$

Interval Matrix and Regularity

Definition

Interval Matrix An interval matrix is the family of matrices

$$\mathbf{A} = \{A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A}\},$$

The midpoint and the radius matrices are defined as

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all interval $m \times n$ matrices is denoted by $\mathbb{IR}^{m \times n}$.

Regularity

$\mathbf{A} \in \mathbb{IR}^{n \times n}$ is regular if each $A \in \mathbf{A}$ is nonsingular.

Interval Matrix and Regularity

Properties

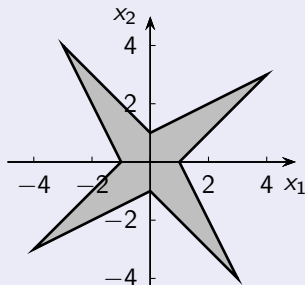
- If $\mathbf{A} \in \mathbb{I}\mathbb{R}^{n \times n}$ is regular, then each realization of $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution and the solution set

$$\{\mathbf{x} \in \mathbb{R}^n : \exists \mathbf{A} \in \mathbf{A} \exists \mathbf{b} \in \mathbf{b} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

is a nonempty and bounded (nonconvex) polyhedron.

Example.

$$\begin{pmatrix} [2, 4] & [-2, 1] \\ [-1, 2] & [2, 4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2, 2] \\ [-2, 2] \end{pmatrix}$$



- Eigenvalues of interval matrices.
- Checking regularity is a co-NP-hard problem (Poljak & Rohn, 1988).

AE Regularity

AE quantification

Disjointly split $\mathbf{A} \in \mathbb{IR}^{n \times n}$ as

$$\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists},$$

where

- \mathbf{A}^{\forall} is the interval matrix comprising universally quantified coefficients,
- \mathbf{A}^{\exists} concerns existentially quantified coefficients.

Example

$$\mathbf{A} = \begin{pmatrix} [-1, 1]^{\forall} & [1, 2]^{\exists} \\ 23 & [5, 6]^{\exists} \end{pmatrix}$$

Definition

An interval matrix $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}$ is called *AE regular* if

$$\forall \mathbf{A}^{\forall} \in \mathbf{A}^{\forall} \exists \mathbf{A}^{\exists} \in \mathbf{A}^{\exists} \text{ such that } A = A^{\forall} + A^{\exists} \text{ is nonsingular.}$$

Quantified Interval Linear Programming

$$\min \mathbf{c}^T \mathbf{x} \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

Traditional Problems

- range of optimal values
- basis stability
- optimal solution set
- robust optimization

AE quantification

- intensively studied with $\forall\exists$ quantification now,
- robust optimization,
- need for $\forall\exists$ quantification ($\sum_i w_i x_i = 1$).

Motivation II.

Definition (AE Solvability of Interval Linear Systems)

$\mathbf{A}^{\forall\exists}x = \mathbf{b}^{\forall\exists}$ is AE solvable if

$$\forall \mathbf{A}^{\forall} \in \mathbf{A}^{\forall}, \forall \mathbf{b}^{\forall} \in \mathbf{b}^{\forall}, \exists \mathbf{A}^{\exists} \in \mathbf{A}^{\exists}, \exists \mathbf{b}^{\exists} \in \mathbf{b}^{\exists} : \\ (\mathbf{A}^{\forall} + \mathbf{A}^{\exists})x = \mathbf{b}^{\forall} + \mathbf{b}^{\exists} \text{ is solvable.}$$

Theorem

If $\mathbf{A}^{\forall\exists}$ is AE regular, then $\mathbf{A}^{\forall\exists}x = \mathbf{b}^{\forall\exists}$ is AE solvable for each $\mathbf{b}^{\forall\exists}$.
The converse is not true in general.

Example

The counter-example for the converse direction is

$$\mathbf{A}^{\forall} = 0, \quad \mathbf{A}^{\exists} = \begin{pmatrix} 0 & [-1, 1] \\ 0 & [-1, 1] \end{pmatrix}.$$

Then $\mathbf{A}^{\forall\exists}$ is not AE regular, but $\mathbf{A}^{\forall\exists}x = \mathbf{b}^{\forall\exists}$ is AE solvable.

AE Regularity

AE Regularity

- is hard to check (generalizes regularity of \mathbf{A}^\forall)
- even regularity of \mathbf{A}^\exists is open problem

Example

Is not AE regular:

$$\mathbf{A}^\exists = \begin{pmatrix} [1, 2] & [1, 2] & [1, 2] & [1, 2] & [1, 2] \\ [1, 2] & 0 & 0 & 0 & [1, 2] \\ [1, 2] & 0 & 0 & 0 & [1, 2] \\ [1, 2] & 0 & 0 & 0 & [1, 2] \\ [1, 2] & [1, 2] & [1, 2] & [1, 2] & [1, 2] \end{pmatrix}.$$

Conjecture

\mathbf{A}^\exists is regular iff it has no submatrix of size $k \times \ell$ that is real and has the rank $k + \ell - n - 1$.

Theorem

\mathbf{A}^\exists is regular iff some vertex matrix $(a_{ij} \in \{\underline{a}_{ij}, \bar{a}_{ij}\})$ is nonsingular.

Open Problems

- Is there a simpler characterization?
- What is the computational complexity?

Interval M-matrix

$A \in \mathbb{R}^{n \times n}$ is an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$.

Theorem (\forall case, Barth & Nuding, 1974)

$\mathbf{A}^\forall \in \mathbb{IR}^{n \times n}$ is an M-matrix iff \underline{A} is an M-matrix and $\bar{A}_{ij} \leq 0$ for $i \neq j$.

Theorem (\exists case)

Let $\mathbf{A}^\exists \in \mathbb{IR}^{n \times n}$ and define $\tilde{A} \in \mathbf{A}^\exists$ as follows

$$\tilde{a}_{ij} = \begin{cases} \bar{a}_{ij} & \text{if } i = j, \\ \arg \min\{|a_{ij}| : a_{ij} \in \mathbf{a}_{ij}^\exists\} & \text{if } i \neq j. \end{cases} \quad (*)$$

Then \mathbf{A}^\exists is an M-matrix iff \tilde{A} is an M-matrix.

Theorem ($\forall\exists$ case)

Denote by $\tilde{A} = \underline{A}^\forall + \tilde{A}^\exists$ the matrix from (*) corresponding to $\underline{A}^\forall + \mathbf{A}^\exists$.

Then $\mathbf{A}^{\forall\exists}$ is AE M-matrix iff \tilde{A} is M-matrix and $(\bar{A}^\forall + \tilde{A}^\exists)_{ij} \leq 0$ for $i \neq j$.

Interval H-matrix

$A \in \mathbb{R}^{n \times n}$ is called an H-matrix, if the comparison matrix $\langle A \rangle$ is an M-matrix, where $\langle A \rangle_{ii} = |a_{ii}|$ and $\langle A \rangle_{ij} = -|a_{ij}|$ for $i \neq j$.

Magnitude and Mignitude

$$\text{mag}(\mathbf{a}) = \max\{|a| : a \in \mathbf{a}\} = |a_c| + a_\Delta,$$

$$\text{mig}(\mathbf{a}) = \min\{|a| : a \in \mathbf{a}\} = \begin{cases} 0 & \text{if } 0 \in \mathbf{a}, \\ \min(|\underline{a}|, |\bar{a}|) & \text{otherwise.} \end{cases}$$

Theorem (\forall case, Neumaier 1984)

Let $\mathbf{A}^\forall \in \mathbb{IR}^{n \times n}$ and define $\tilde{A} \in \mathbf{A}^\forall$ as follows

$$\tilde{a}_{ij} = \begin{cases} \text{mig}(\mathbf{a}_{ij}^\forall) & \text{if } i = j, \\ -\text{mag}(\mathbf{a}_{ij}^\forall) & \text{if } i \neq j. \end{cases}$$

Then \mathbf{A}^\forall is an H-matrix iff \tilde{A} is an M-matrix.

Theorem (\exists case)

Let $\mathbf{A}^\exists \in \mathbb{IR}^{n \times n}$ and define $\tilde{\mathbf{A}} \in \mathbf{A}^\exists$ as follows

$$\tilde{a}_{ij} = \begin{cases} \text{mag}(\mathbf{a}_{ij}^\exists) & \text{if } i = j, \\ -\text{mig}(\mathbf{a}_{ij}^\exists) & \text{if } i \neq j. \end{cases}$$

Then \mathbf{A}^\exists is an H-matrix iff $\tilde{\mathbf{A}}$ is an M-matrix.

Theorem ($\exists \forall$ case)

Define the matrix $\tilde{\mathbf{A}}$ as follows

$$\tilde{a}_{ij}^\exists = \begin{cases} \text{mig}(\mathbf{a}_{ij}^\forall) + \text{mag}(\mathbf{a}_{ij}^\exists) & \text{if } i = j, \\ -\text{mag}(\mathbf{a}_{ij}^\forall) - \text{mig}(\mathbf{a}_{ij}^\exists) & \text{if } i \neq j. \end{cases}$$

Then $\mathbf{A}^{\forall \exists}$ is an AE H-matrix if and only if $\tilde{\mathbf{A}}$ is an M-matrix.

Theorem (Neumaier, 1990)

If A_c^\forall is an M-matrix, then A^\forall is regular iff A^\forall is H-matrix.

Example

For AE quantification, this statement is no longer valid.

$$A^{\forall\exists} = \begin{pmatrix} [0.8, 1]^\exists & -[0, 1]^\forall \\ -1 & 1 \end{pmatrix}.$$

Then $A_c^{\forall\exists}$ is an M-matrix and $A^{\forall\exists}$ is AE regular, but not an AE H-matrix.

Structured Quantifiers Position

Theorem

The square interval matrix

$$\mathbf{A}^{\forall\exists} = \begin{pmatrix} \mathbf{B}^{\forall} & \mathbf{b}^{\exists} \\ \mathbf{C}^{\forall} & \mathbf{c}^{\forall} \end{pmatrix}, \text{ where } b^{\exists}_{\Delta} > 0,$$

is AE regular iff $(\mathbf{B}^T \ \mathbf{C}^T)^{\forall}$ and $(\mathbf{C} \ \mathbf{c})^{\forall}$ have full row rank.

Theorem

Let $\mathbf{B} \in \mathbb{R}^{n \times k}$ and $\mathbf{C} \in \mathbb{R}^{n \times (n-k)}$ with $C_{\Delta} > 0$. Then $(\mathbf{B}^{\forall} \ \mathbf{C}^{\exists})$ is AE regular iff \mathbf{B}^{\forall} has full column rank.

Conjecture

The square interval matrix

$$\begin{pmatrix} \mathbf{B}^{\forall} & \mathbf{D}^{\exists} \\ \mathbf{C}^{\forall} & \mathbf{E}^{\forall} \end{pmatrix}, \text{ where } D^{\exists}_{\Delta} > 0,$$

is AE regular iff $(\mathbf{B}^T \ \mathbf{C}^T)^{\forall}$ and $(\mathbf{C} \ \mathbf{E})^{\forall}$ have full row rank.

Summary

- We introduced a generalized regularity concept of interval matrices.
- Easy to handle cases:
 - M-matrices
 - H-matrices
 - special position of quantifiers

Many Open Problems

- AE regularity of $\mathbf{A}^{\forall\exists}$: characterization, ...
- regularity of \mathbf{A}^{\exists} : complexity, characterization, ...
- structured quantifiers position conjecture
- identifying other polynomial cases