Introduction to Interval Computation and Numerical Verification

part I.

Milan Hladík

Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

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Outline

Next Section

Interval Computation

What is interval computation

Solving problems with interval data
(or using interval techniques for non-interval problems)

What is **not** interval computation

- stochastic computation
- fuzzy computation

Interval paradigm

Take into account all possible realizations rigorously.

Where interval data do appear

1. numerical analysis (handling rounding errors)
2. computer-assisted proofs
3. global optimization
4. modelling uncertainty

Numerical Analysis

Example (Rump, 1988)

Consider the expression

$$f = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b},$$

with

$$a = 77617, \quad b = 33096.$$

Calculations from 80s gave

$$\begin{array}{rl} \text{single precision} & f \approx 1.172603\ldots \\ \text{double precision} & f \approx 1.1726039400531\ldots \\ \text{extended precision} & f \approx 1.172603940053178\ldots \\ \text{the true value} & f = -0.827386\ldots \end{array}$$

Computer-Assisted Proofs

Kepler conjecture

What is the densest packing of balls? (Kepler, 1611)

That one how the oranges are stacked in a shop.

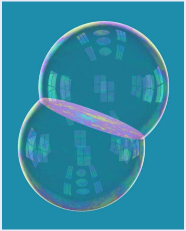
The conjecture was proved by T.C. Hales (2005).

Double bubble problem

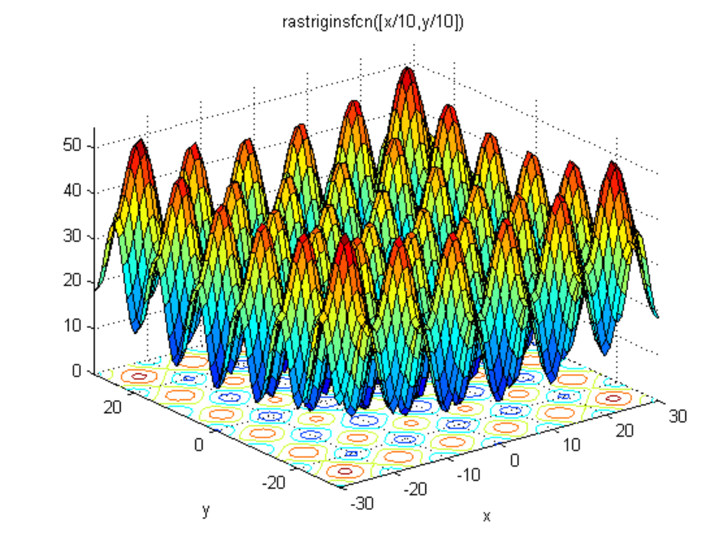
What is the minimal surface of two given volumes?

Two pieces of spheres meeting at an angle of 120°.

Hass and Schlafly (2000) proved the equally sized case. Hutchings et al. (2002) proved the general case.



Rastrigin's function $f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))$



Further Sources of Intervals

- Mass number of chemical elements (sue to several stable isotopes)
  - $[12.0096, 12.0116]$ for the carbon
- physical constants
  - $[9.78,\ 9.82]\, ms^{-2}$ for the gravitational acceleration
- mathematical constants
  - $\pi \in [3.1415926535897932384,\ 3.1415926535897932385]$.
- measurement errors
  - temperature measured $23°C \pm 1°C$
- discretization
  - time is split in days
  - temperature during the day in $[-8, 3]°C$ for Ostrava in January
- missing data
  - What was the temperature in Ostrava on January 31, 1999?
  - Very probably in $[-25, 15]°C$.
- processing a state space
  - find robot singularities, where it may breakdown
  - check joint angles $[0, 180]°$.

Next Section

Notation

An interval matrix

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A^c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $m \times n$ interval matrices: $\mathbb{IR}^{m \times n}$.

Main problem

Let $f \colon \mathbb{R}^n \mapsto \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{IR}^n$. Determine the image

$$f(\boldsymbol{x}) = \{f(x) : x \in \boldsymbol{x}\}.$$

Interval Arithmetic

Interval arithmetic (incl. rounding, IEEE standard)

$$\begin{aligned}
\boldsymbol{a} + \boldsymbol{b} &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
\boldsymbol{a} - \boldsymbol{b} &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
\boldsymbol{a} \cdot \boldsymbol{b} &= [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})], \\
\boldsymbol{a}/\boldsymbol{b} &= [\min(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}), \max(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b})], \quad 0 \notin \boldsymbol{b}.
\end{aligned}$$

Theorem (Basic properties of interval arithmetic)

- *Interval addition and multiplication is commutative and associative.*
- *It is not distributive in general, but sub-distributive instead,*

$$\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{IR} : \boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) \subseteq \boldsymbol{ab} + \boldsymbol{ac}.$$

Example ($\boldsymbol{a} = [1, 2]$, $\boldsymbol{b} = 1$, $\boldsymbol{c} = -1$)

$$\begin{aligned}
\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) &= [1, 2] \cdot (1 - 1) = [1, 2] \cdot 0 = 0, \\
\boldsymbol{ab} + \boldsymbol{ac} &= [1, 2] \cdot 1 + [1, 2] \cdot (-1) = [1, 2] - [1, 2] = [-1, 1].
\end{aligned}$$

Next Section

Images of Functions

Monotone functions

If $f : \boldsymbol{x} \to \mathbb{R}$ is non-decreasing, then $f(\boldsymbol{x}) = [f(\underline{x}), f(\overline{x})]$.

Example

$\exp(\boldsymbol{x}) = [\exp(\underline{x}), \exp(\overline{x})]$, $\log(\boldsymbol{x}) = [\log(\underline{x}), \log(\overline{x})]$, ...

Some basic functions

Images $\boldsymbol{x}^2$, $\sin(\boldsymbol{x})$, ..., are easily calculated, too.

$$\boldsymbol{x}^2 = \begin{cases} [\min(\underline{x}^2, \overline{x}^2), \max(\underline{x}^2, \overline{x}^2)] & \text{if } 0 \notin \boldsymbol{x}, \\ \boldsymbol{x}^2 = [0, \max(\underline{x}^2, \overline{x}^2)] & \text{otherwise} \end{cases}$$

But...

...what to do for more complex functions?

Images of Functions

Notice

$f(\boldsymbol{x})$ need not be an interval (neither closed nor connected).

Interval hull $\square f(x)$

Compute the interval hull instead

$$\square f(\boldsymbol{x}) = \bigcap_{\boldsymbol{v} \in \mathbb{IR}^n \,:\, f(\boldsymbol{x}) \subseteq \boldsymbol{v}} \boldsymbol{v}.$$

Bad news

Computing $\square f(\boldsymbol{x})$ is still very difficult (NP-hard, undecidable).

Interval enclosure

Compute as tight as possible $\boldsymbol{v} \in \mathbb{IR}^n : f(\boldsymbol{x}) \subseteq \boldsymbol{v}$.

Interval Functions

Definition (Inclusion isotonicity)

$\boldsymbol{f} : \mathbb{IR}^n \mapsto \mathbb{IR}$ is *inclusion isotonic* if for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}^n$:

$$\boldsymbol{x} \subseteq \boldsymbol{y} \Rightarrow \boldsymbol{f}(\boldsymbol{x}) \subseteq \boldsymbol{f}(\boldsymbol{y}).$$

Definition (Interval extension)

$\boldsymbol{f} : \mathbb{IR}^n \mapsto \mathbb{IR}$ is *an interval extension* of $f : \mathbb{R}^n \mapsto \mathbb{R}$ if for every $x \in \mathbb{R}^n$:

$$f(x) = \boldsymbol{f}(x).$$

Theorem (Fundamental theorem of interval analysis)

If $\boldsymbol{f} : \mathbb{IR}^n \mapsto \mathbb{IR}$ *satisfies both properties, then*

$$f(\boldsymbol{x}) \subseteq \boldsymbol{f}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{IR}^n.$$

Proof.

For every $x \in \boldsymbol{x}$, one has by interval extension and inclusion isotonicity that $f(x) = \boldsymbol{f}(x) \subseteq \boldsymbol{f}(\boldsymbol{x})$, whence $f(\boldsymbol{x}) \subseteq \boldsymbol{f}(\boldsymbol{x})$. □

Natural Interval Extension

Definition (Natural interval extension)

Let $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ be a function given by an arithmetic expression. The corresponding *natural interval extension* $\boldsymbol{f}$ of f is defined by that expression when replacing real arithmetic by the interval one.

Theorem

Natural interval extension of an arithmetic expression is both an interval extension and inclusion isotonic.

Proof.

It is easy to see that interval arithmetic is both an interval extension and inclusion isotonic. Next, proceed by mathematical induction. □

Natural Interval Extension

Example

$$f(x) = x^2 - x, \quad x \in \boldsymbol{x} = [-1, 2].$$

Then

$$\boldsymbol{x}^2 - \boldsymbol{x} = [-1, 2]^2 - [-1, 2] = [-2, 5],$$
$$\boldsymbol{x}(\boldsymbol{x} - 1) = [-1, 2]([-1, 2] - 1) = [-4, 2],$$
$$\text{Best one?} (\boldsymbol{x} - \tfrac{1}{2})^2 - \tfrac{1}{4} = ([-1, 2] - \tfrac{1}{2})^2 - \tfrac{1}{4} = [-\tfrac{1}{4}, 2].$$

Theorem

Suppose that in an expression of $f : \mathbb{R}^n \mapsto \mathbb{R}$ each variable $x_1, \dots, x_n$ appears at most once. The corresponding natural interval extension $\boldsymbol{f}(\boldsymbol{x})$ satisfies for every $\boldsymbol{x} \in \mathbb{IR}^n$: $f(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$.

Proof.

Inclusion "$\subseteq$" by the previous theorems.
Inclusion "$\supseteq$" by induction and exactness of interval arithmetic. $\square$

Software

Matlab/Octave libraries

- *Intlab* (by S.M. Rump),
  interval arithmetic and elementary functions
  `http://www.ti3.tu-harburg.de/~rump/intlab/`
- *Versoft* (by J. Rohn),
  verification software written in Intlab
  `http://uivtx.cs.cas.cz/~rohn/matlab/`
- *Lime* (by M. Hladík, J. Horáček et al.),
  interval methods written in Intlab, under development
  `http://kam.mff.cuni.cz/~horacek/projekty/lime/`

Other languages libraries

- *Int4Sci Toolbox* (by Coprin team, INRIA),
  A Scilab Interface for Interval Analysis
  `http://www-sop.inria.fr/coprin/logiciels/Int4Sci/`
- *C++ libraries*: C-XSC, PROFIL/BIAS, BOOST interval, FILIB++,...
- *many others*: for Fortran, Pascal, Lisp, Maple, Mathematica,...

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Next Section

Solution Set

Interval linear equations

Let $\boldsymbol{A} \in \mathbb{IR}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{IR}^{m}$. The family of systems

$$Ax = b, \quad A \in \boldsymbol{A},\ b \in \boldsymbol{b}.$$

is called interval linear equations and abbreviated as $\boldsymbol{A}x = \boldsymbol{b}$.

Solution set

The solution set is defined

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \boldsymbol{A} \exists b \in \boldsymbol{b} : Ax = b\}.$$

Important notice

We do not want to compute $\boldsymbol{x} \in \mathbb{IR}^n$ such that $\boldsymbol{Ax} = \boldsymbol{b}$.

Theorem (Oettli–Prager, 1964)

The solution set Σ is a non-convex polyhedral set described by

$$|A^c x - b^c| \leq A^\Delta |x| + b^\Delta.$$

Proof of Oettli–Prager Theorem ($|A^c x - b^c| \le A^\Delta |x| + b^\Delta$)

Let $x \in \Sigma$, that is, $Ax = b$ for some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Now,

$$\begin{aligned}|A^c x - b^c| &= |(A^c - A)x + (Ax - b) + (b - b^c)| = |(A^c - A)x + (b - b^c)| \\ &\le |A^c - A||x| + |b - b^c| \le A^\Delta |x| + b^\Delta.\end{aligned}$$

Conversely, let $x \in \mathbb{R}^n$ satisfy the inequalities. Define $y \in [-1, 1]^m$ as

$$y_i = \begin{cases} \frac{(A^c x - b^c)_i}{(A^\Delta |x| + b^\Delta)_i} & \text{if } (A^\Delta |x| + b^\Delta)_i > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Now, we have $(A^c x - b^c)_i = y_i (A^\Delta |x| + b^\Delta)_i$, or,

$$A^c x - b^c = \operatorname{diag}(y)(A^\Delta |x| + b^\Delta).$$

Define $z := \operatorname{sgn}(x)$, then $|x| = \operatorname{diag}(z)x$ and we can write

$$A^c x - b^c = \operatorname{diag}(y) A^\Delta \operatorname{diag}(z) x + \operatorname{diag}(y) b^\Delta,$$

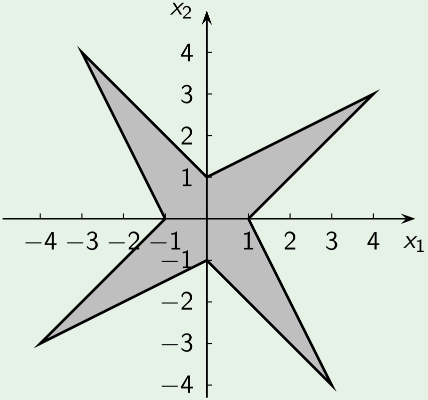
or

$$(A^c - \operatorname{diag}(y) A^\Delta \operatorname{diag}(z))x = b^c + \operatorname{diag}(y) b^\Delta. \qquad \square$$

Example of the Solution Set

Example (Barth & Nuding, 1974))

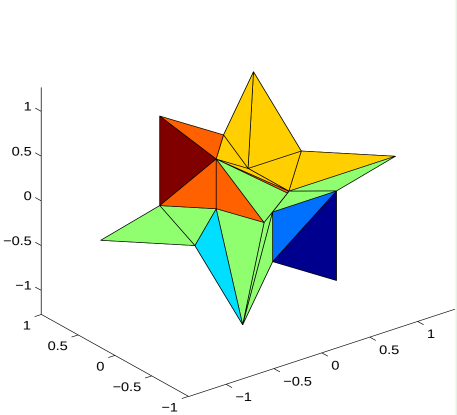
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$



Example of the Solution Set

Example

$$\begin{pmatrix} [3,5] & [1,3] & -[0,2] \\ -[0,2] & [3,5] & [0,2] \\ [0,2] & -[0,2] & [3,5] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}.$$



Topology of the Solution Set

Proposition

In each orthant, Σ is either empty or a convex polyhedral set.

Proof.

Restriction to the orthant given by $s \in \{\pm 1\}^n$:

$$|A^c x - b^c| \le A^\Delta |x| + b^\Delta, \ \operatorname{diag}(s)x \ge 0.$$

Since $|x| = \operatorname{diag}(s)x$, we have

$$|A^c x - b^c| \le A^\Delta \operatorname{diag}(s)x + b^\Delta, \ \operatorname{diag}(s)x \ge 0.$$

Using $|a| \le b \ \Leftrightarrow a \le b, \ -a \le b$, we get

$$(A^c - A^\Delta \operatorname{diag}(s))x \le \overline{b}, \ (-A^c - A^\Delta \operatorname{diag}(s))x \le -\underline{b}, \ \operatorname{diag}(s)x \ge 0. \quad \square$$

Corollary

The solutions of $\boldsymbol{A}x = \boldsymbol{b}$, $x \ge 0$ is described by $\underline{A}x \le \overline{b}, \ \overline{A}x \ge \underline{b}, \ x \ge 0$.

Remark

Checking $\Sigma \neq \emptyset$ and boundedness are NP-hard.

Interval Hull $\square\Sigma$

Goal

Seeing that Σ is complicated, compute $\square\Sigma$ instead.

First idea

Go through all 2^n orthants of $\mathbb{R}^n$, determine interval hull of restricted sets (by solving $2n$ linear programs), and then put together.

Theorem

If $\boldsymbol{A}$ is regular (each $A \in \boldsymbol{A}$ is nonsingular), Σ is bounded and connected.

Theorem (Jansson, 1997)

When $\Sigma \neq \emptyset$, then exactly one of the following alternatives holds true:

1. *Σ is bounded and connected.*
2. *Each topologically connected component of Σ is unbounded.*

Second idea – Jansson's algorithm

Check the orthant with $(A^c)^{-1}b^c$ and then all the topologically connected.

Polynomial Cases

Two basic polynomial cases

1. $A^c = I_n$,
2. $\boldsymbol{A}$ is inverse nonnegative, i.e., $A^{-1} \geq 0 \ \forall A \in \boldsymbol{A}$.

Theorem (Kuttler, 1971)

$\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ is inverse nonnegative if and only if $\underline{A}^{-1} \geq 0$ and $\overline{A}^{-1} \geq 0$.

Theorem

Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be inverse nonnegative. Then

1. $\square\Sigma = [\overline{A}^{-1}\underline{b}, \underline{A}^{-1}\overline{b}]$ *when* $\underline{b} \geq 0$,
2. $\square\Sigma = [\underline{A}^{-1}\underline{b}, \overline{A}^{-1}\overline{b}]$ *when* $\underline{b} \leq 0$,
3. $\square\Sigma = [\underline{A}^{-1}\underline{b}, \underline{A}^{-1}\overline{b}]$ *when* $0 \in \boldsymbol{b}$.

Proof.

1. Let $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Since $\overline{b} \geq b \geq \underline{b} \geq 0$ and $\underline{A}^{-1} \geq A^{-1} \geq \overline{A}^{-1} \geq 0$, we get $\overline{A}^{-1}\underline{b} \leq A^{-1}b \leq \underline{A}^{-1}\overline{b}$.

□

Next Section

Preconditioning

Enclosure

Since Σ is hard to determine and deal with, we seek for enclosures

$$\boldsymbol{x} \in \mathbb{IR}^n \text{ such that } \Sigma \subseteq \boldsymbol{x}.$$

Many methods for enclosures exists, usually employ preconditioning.

Preconditioning (Hansen, 1965)

Let $C \in \mathbb{R}^{n \times n}$. The preconditioned system of equations:

$$(C\boldsymbol{A})x = C\boldsymbol{b}.$$

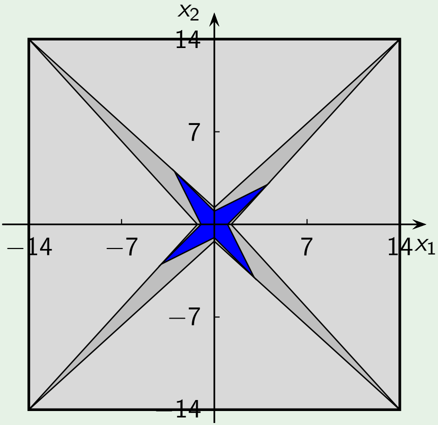
Remark

- the solution set of the preconditioned systems contains Σ
- usually, we use $C \approx (A^c)^{-1}$, which is best in some sense
- then we can compute the best enclosure (Hansen, 1992, Bliek, 1992, Rohn, 1993)

Preconditioning

Example (Barth & Nuding, 1974))

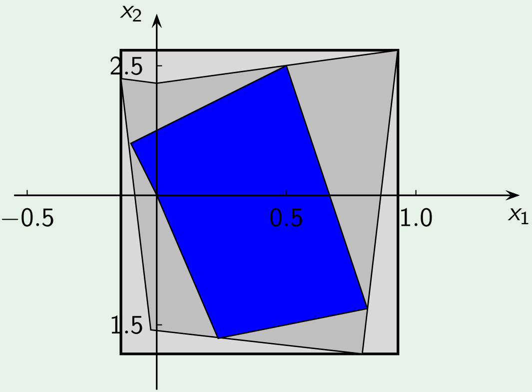
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$



Preconditioning

Example (typical case)

$$\begin{pmatrix} [6,7] & [2,3] \\ [1,2] & -[4,5] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [6,8] \\ -[7,9] \end{pmatrix}$$



Interval Gaussian Elimination

Interval Gaussian elimination = Gaussian elimination + interval arithmetic.

Example (Barth & Nuding, 1974))

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$

Then we proceed as follows

$$\begin{pmatrix} [2,4] & [-2,1] & [-2,2] \\ [-1,2] & [2,4] & [-2,2] \end{pmatrix} \sim \begin{pmatrix} [2,4] & [-2,1] & [-2,2] \\ 0 & [1,6] & [-4,4] \end{pmatrix}.$$

By back substitution, we compute

$$\begin{aligned} \boldsymbol{x}_2 &= [-4,4], \\ \boldsymbol{x}_1 &= \left([-2,2] - [-2,1] \cdot [-4,4]\right) / [2,4] = [-5,5]. \end{aligned}$$

Interval Jacobi and Gauss-Seidel Iterations

Idea

From the ith equation of $Ax = b$ we get

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j\right).$$

If $\boldsymbol{x}^0 \supseteq \Sigma$ is an initial enclosure, then

$$x_i \in \frac{1}{\boldsymbol{a}_{ii}}\left(\boldsymbol{b}_i - \sum_{j\neq i} \boldsymbol{a}_{ij}\boldsymbol{x}_j^0\right), \quad \forall x \in \Sigma.$$

Thus, we can tighten the enclosure by iterations

Interval Jacobi / Gauss–Seidel iterations ($k = 1, 2, \dots$)

1: **for** $i = 1, \dots, n$ **do**
2: $\quad \boldsymbol{x}_i^k := \frac{1}{\boldsymbol{a}_{ii}}\left(\boldsymbol{b}_i - \sum_{j\neq i} \boldsymbol{a}_{ij}\boldsymbol{x}_j^{k-1}\right) \cap \boldsymbol{x}_i^{k-1};$
3: **end for**

Krawczyk Iterations

Krawczyk operator

Krawczyk operator $K\colon \mathbb{IR}^n \to \mathbb{IR}^n$ reads

$$K(\boldsymbol{x}) := C\boldsymbol{b} + (I_n - C\boldsymbol{A})\boldsymbol{x}$$

Proposition

If $x \in \boldsymbol{x} \cap \Sigma$, then $x \in K(\boldsymbol{x})$.

Proof.

Let $x \in \boldsymbol{x} \cap \Sigma$, so $Ax = b$ for some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Thus $CAx = Cb$, whence $x = Cb + (I_n - CA)x \in C\boldsymbol{b} + (I_n - C\boldsymbol{A})\boldsymbol{x} = K(\boldsymbol{x})$. □

Krawczyk iterations

Let $\boldsymbol{x}^0 \supseteq \Sigma$ is an initial enclosure, and iterate $(k = 1, 2, \dots)$:

1: $\boldsymbol{x}^k := K(\boldsymbol{x}^{k-1}) \cap \boldsymbol{x}^{k-1}$;

ε-inflation

Theorem

Let $\boldsymbol{x} \in \mathbb{IR}^n$ *and* $C \in \mathbb{R}^{n\times n}$. *If*

$$K(\boldsymbol{x}) = C\boldsymbol{b} + (I - C\boldsymbol{A})\boldsymbol{x} \subseteq int\, \boldsymbol{x},$$

then C *is nonsingular,* $\boldsymbol{A}$ *is regular, and* $\Sigma \subseteq \boldsymbol{x}$.

Proof.

Existence of a solution based on Brouwer's fixed-point theorem.
Nonsingularity and uniqueness based on the Perron–Frobenius theory. □

Remark

- A reverse iteration method to the Krawczyk method.
- It starts with a small box around $(A^c)^{-1}b^c$, and then iteratively inflates the box.
- Implemented in Intlab v. 6.

Next Section

Regularity

Definition (Regularity)

$\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is regular if each $A \in \boldsymbol{A}$ is nonsingular.

Theorem

Checking regularity of an interval matrix is co-NP-hard.

Forty necessary and sufficient conditions for regularity of $\boldsymbol{A}$ by Rohn (2010):

1. The system $|A^c x| \leq A^\Delta |x|$ has the only solution $x = 0$.
2. $\det(A^c - \operatorname{diag}(y) A^\Delta \operatorname{diag}(z))$ is constantly either positive or negative for each $y, z \in \{\pm 1\}^n$.
3. For each $y \in \{\pm 1\}^n$, the system $A^c x - \operatorname{diag}(y) A^\Delta |x| = y$ has a solution.
4. ...

Regularity – Sufficient / Necessary Conditions

Theorem (Beeck, 1975)

If $\rho(|(A^c)^{-1}|A^\Delta) < 1$, *then* $\boldsymbol{A}$ *is regular.*

Proof.

Precondition $\boldsymbol{A}$ by the midpoint inverse: $\boldsymbol{M} := (A^c)^{-1}\boldsymbol{A}$. Now,

$$M^c = I_n, \quad M^\Delta = |(A^c)^{-1}|A^\Delta,$$

and for each $M \in \boldsymbol{M}$ we have

$$|M - M^c| = |M - I_n| \le M^\Delta.$$

From the theory of eigenvalues of nonnegative matrices it follows

$$\rho(M - I_n) \le \rho(M^\Delta) < 1,$$

so M has no zero eigenvalue and is nonsingular. $\square$

Necessary condition

If $0 \in \boldsymbol{A}x$ for some $0 \ne x \in \mathbb{R}^n$, then $\boldsymbol{A}$ is not regular. (Try $x := (A^c)^{-1}_{*i}$)

Next Section

Parametric Interval Systems

Parametric interval systems

$$A(p)x = b(p),$$

where the entries of $A(p)$ and $b(p)$ depend on parameters $p_1 \in \boldsymbol{p}_1, \ldots, p_K \in \boldsymbol{p}_K$.

Definition (Solution set)

$$\Sigma_{\mathrm{p}} = \{x \in \mathbb{R}^n \colon A(p)x = b(p) \text{ for some } p \in \boldsymbol{p}\}.$$

Relaxation

Compute (enclosures of) the ranges $\boldsymbol{A} := A(\boldsymbol{p})$ and $\boldsymbol{b} := b(\boldsymbol{p})$ and solve

$$\boldsymbol{A}x = \boldsymbol{b}.$$

May overestimate a lot!

Special Case: Parametric Linear Interval Systems

Parametric linear interval systems

$$A(p)x = b(p),$$

where

$$A(p) = \sum_{k=1}^{K} A_k p_k, \quad b(p) = \sum_{k=1}^{K} b_k p_k$$

and $p \in \boldsymbol{p}$ for some given interval vector $\boldsymbol{p} \in \mathbb{IR}^K$, matrices $A_1, \ldots, A_K \in \mathbb{R}^{n \times n}$ and vectors $b_1, \ldots, b_n \in \mathbb{R}^n$.

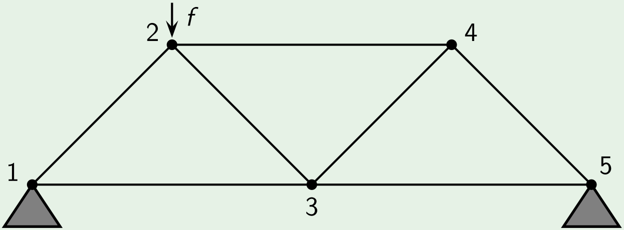
Remark

It covers many structured matrices: symmetric, skew-symmetric, Toeplitz or Hankel.

Parametric Linear Interval Systems – Example

Example (Displacements of a truss structure (Skalna, 2006))

The 7-bar truss structure subject to downward force.
The stiffnesses s_{ij} of bars are uncertain.
The displacements d of the nodes, are solutions of the system $Kd = f$, where f is the vector of forces.



Parametric Linear Interval Systems – Example

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The 7-bar truss structure subject to downward force.
The stiffnesses s_{ij} of bars are uncertain.
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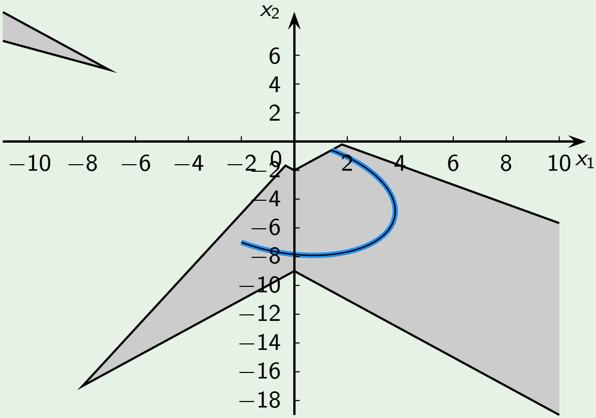
$$K = \begin{pmatrix}
\frac{s_{12}}{2} + s_{13} & -\frac{s_{12}}{2} & -\frac{s_{12}}{2} & -s_{13} & 0 & 0 & 0 \\
-\frac{s_{21}}{2} & \frac{s_{21}+s_{23}}{2} + s_{24} & \frac{s_{21}-s_{23}}{2} & -\frac{s_{23}}{2} & \frac{s_{23}}{2} & -s_{24} & 0 \\
-\frac{s_{21}}{2} & \frac{s_{21}-s_{23}}{2} & \frac{s_{21}+s_{23}}{2} & \frac{s_{23}}{2} & -\frac{s_{23}}{2} & 0 & 0 \\
-s_{31} & -\frac{s_{32}}{2} & \frac{s_{32}}{2} & s_{31} + \frac{s_{32}+s_{34}}{2} + s_{35} & \frac{s_{34}-s_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\
0 & \frac{s_{32}}{2} & -\frac{s_{32}}{2} & \frac{s_{34}-s_{32}}{2} & \frac{s_{34}+s_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\
0 & -s_{42} & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & s_{42} + \frac{s_{43}+s_{45}}{2} & 0 \\
0 & 0 & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & 0 & \frac{s_{43}+s_{45}}{2}
\end{pmatrix}$$

Parametric Linear Interval Systems – Example

Example

$$\begin{pmatrix} 1-2p & 1 \\ 2 & 4p-1 \end{pmatrix} x = \begin{pmatrix} 7p-9 \\ 3-2p \end{pmatrix}, \quad p \in \boldsymbol{p} = [0,1].$$



Parametric Linear Interval Systems – Solution Set

Theorem

If $x \in \Sigma_\mathrm{p}$, then it solves

$$|A(p^c)x - b(p^c)| \leq \sum_{k=1}^{K} p_k^\Delta |A^k x - b^k|.$$

Proof.

$$|A(p^c)x - b(p^c)| = \left| \sum_{k=1}^{K} p_k^c (A^k x - b^k) \right| = \left| \sum_{k=1}^{K} p_k^c (A^k x - b^k) - \sum_{k=1}^{K} p_k (A^k x - b^k) \right|$$

$$= \left| \sum_{k=1}^{K} (p_k^c - p_k)(A^k x - b^k) \right| \leq \sum_{k=1}^{K} |p_k^c - p_k| |A^k x - b^k| \leq \sum_{k=1}^{K} p_k^\Delta |A^k x - b^k|. \qquad \square$$

- Popova (2009) showed that it is the complete characterization of Σ_p as long as no interval parameter appears in more than one equation.
- Checking $x \in \Sigma_\mathrm{p}$ for a given $x \in \mathbb{R}^n$ is a polynomial problem via linear programming.

Parametric Linear Interval Systems – Enclosures

Relaxation and preconditioning – First idea

Evaluate $\boldsymbol{A} := A(\boldsymbol{p})$, $\boldsymbol{b} := b(\boldsymbol{p})$, choose $C \in \mathbb{R}^{n\times n}$ and solve

$$(C\boldsymbol{A})x = C\boldsymbol{b}.$$

Relaxation and preconditioning – Second idea

Solve $\boldsymbol{A}'x = \boldsymbol{b}'$, where

$$\boldsymbol{A}' := \sum_{k=1}^{K}(CA^k)\boldsymbol{p}_k, \quad \boldsymbol{b}' := \sum_{k=1}^{K}(Cb^k)\boldsymbol{p}_k.$$

Second idea is provably better

Due to sub-distributivity law,

$$\boldsymbol{A}' := \sum_{k=1}^{K}(CA^k)\boldsymbol{p}_k \subseteq C\left(\sum_{k=1}^{K}A^k\boldsymbol{p}_k\right) = (C\boldsymbol{A}).$$

Special Case: Symmetric Systems

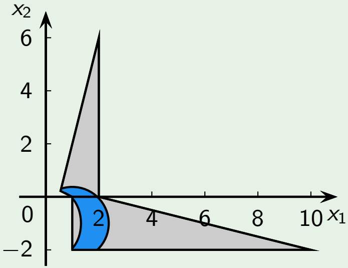
The symmetric solution set of $\boldsymbol{A}x = \boldsymbol{b}$

$$\{x \in \mathbb{R}^n \colon Ax = b \text{ for some symmetric } A \in \boldsymbol{A} \text{ and } b \in \boldsymbol{b}\}.$$

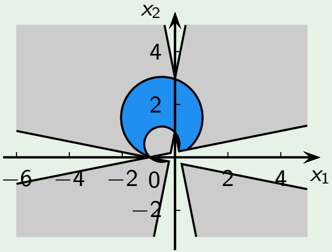
Described by $\frac{1}{2}(4^n - 3^n - 2 \cdot 2^n + 3) + n$ nonlinear inequalities (H., 2008).

Example

$\boldsymbol{A} = \begin{pmatrix} [1,2] & [0,a] \\ [0,a] & -1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$



$\boldsymbol{A} = \begin{pmatrix} -1 & [-5,5] \\ [-5,5] & 1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 1 \\ [1,3] \end{pmatrix}.$



Application: Least Square Solutions

Least square solution

Let $A \in \mathbb{IR}^{m\times n}$, $b \in \mathbb{IR}^m$ and $m > n$. The least square solution of

$$Ax = b,$$

is defined as the optimal solution of

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2,$$

or, alternatively as the solution to

$$A^T Ax = A^T b.$$

Interval least square solution set

Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{IR}^m$ and $m > n$. The LSQ solution set is defined

$$\Sigma_{LSQ} := \{x \in \mathbb{R}^n : \exists A \in \boldsymbol{A} \exists b \in \boldsymbol{b} : A^T Ax = A^T b\}.$$

Proposition

Σ_{LSQ} is contained in the solution set to $\boldsymbol{A}^T\boldsymbol{A}x = \boldsymbol{A}^T\boldsymbol{b}$.

Application: Least Square Solutions

Proposition

Σ_{LSQ} is contained in the solution set to

$$\begin{pmatrix} 0 & \boldsymbol{A}^T \\ \boldsymbol{A} & I_m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix}. \tag{1}$$

Proof.

Let $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$. If x, y solve

$$A^T y = 0, \; Ax + y = b,$$

then

$$0 = A^T(b - Ax) = A^T b - A^T Ax,$$

and vice versa. $\square$

Proposition

Relaxing the dependencies, the solution set to $\boldsymbol{A}^T \boldsymbol{A} x = \boldsymbol{A}^T \boldsymbol{b}$ is contained in the solution set to (1).