Generalized Regularity and Singularity of Interval Matrices

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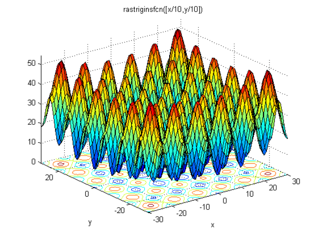
Interval Matrices – Motivation

Where interval data do appear

- numerical analysis (handling rounding errors)
  - $\frac{1}{3} \in [0.33333333333333, 0.33333333333334]$
  - $\pi \in [3.1415926535897932384, 3.1415926535897932385]$.
- constraint solving and global optimization
  - find robot singularities, where it may breakdown
    check joint angles $[0, 180]^\circ$
  - find minimum of $f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))$



- modelling uncertainty
  - temperature measured in Da Nang $29^\circ\mathrm{C} \pm 1^\circ\mathrm{C}$

Interval Matrix and Regularity

Definition

Interval Matrix An interval matrix is the family of matrices

$$\boldsymbol{A} = \{A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A}\},$$

The midpoint and the radius matrices are defined as

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all interval $m \times n$ matrices is denoted by $\mathbb{IR}^{m \times n}$.

Regularity

$\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ is regular if each $A \in \boldsymbol{A}$ is nonsingular.

Interval Matrix and Regularity

Properties

- If $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is regular, then each realization of $\boldsymbol{A}x = \boldsymbol{b}$ has a unique solution and the solution set

$$\{x \in \mathbb{R}^n : \exists A \in \boldsymbol{A} \, \exists b \in \boldsymbol{b} : Ax = b\}$$

  is a nonempty and bounded (nonconvex) polyhedron.
- Checking regularity is a co-NP-hard problem (Poljak & Rohn, 1988).

AE Regularity

AE quantification

Disjointly split $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ as

$$\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists},$$

where

- $\boldsymbol{A}^{\forall}$ is the interval matrix comprising universally quantified coefficients,
- $\boldsymbol{A}^{\exists}$ concerns existentially quantified coefficients.

Example

$$\boldsymbol{A} = \begin{pmatrix} [-1,1]^{\forall} & [1,2]^{\exists} \\ 23 & [5,6]^{\exists} \end{pmatrix}$$

Definition

An interval matrix $\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$ is called *AE regular* if

$$\forall A^{\forall} \in \boldsymbol{A}^{\forall}\, \exists A^{\exists} \in \boldsymbol{A}^{\exists} \text{ such that } A = A^{\forall} + A^{\exists} \text{ is nonsingular.}$$

Motivation I.

Quantified Interval Linear Programming

$$\min \ \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x = \boldsymbol{b}, \ x \geq 0$$

Traditional Problems

- range of optimal values
- basis stability
- optimal solution set
- robust optimization

AE quantification

- intensively studied with $\forall\exists$ quantification now,
- robust optimization,
- need for $\forall\exists$ quantification ($\sum_i w_i x_i = 1$).

Motivation II.

Definition (AE Solvability of Interval Linear Systems)

$\boldsymbol{A}^{\forall\exists}x = \boldsymbol{b}^{\forall\exists}$ is *AE solvable* if

$$\forall A^{\forall} \in \boldsymbol{A}^{\forall}, \forall b^{\forall} \in \boldsymbol{b}^{\forall}, \exists A^{\exists} \in \boldsymbol{A}^{\exists}, \exists b^{\exists} \in \boldsymbol{b}^{\exists} :$$

$$(A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists} \text{ is solvable.}$$

Theorem

If $\boldsymbol{A}^{\forall\exists}$ is AE regular, then $\boldsymbol{A}^{\forall\exists}x = \boldsymbol{b}^{\forall\exists}$ is AE solvable for each $\boldsymbol{b}^{\forall\exists}$. The converse is not true in general.

Example

The counter-example for the converse direction is

$$\boldsymbol{A}^{\forall} = 0, \quad \boldsymbol{A}^{\exists} = \begin{pmatrix} 0 & [-1, 1] \\ 0 & [-1, 1] \end{pmatrix}.$$

Then $\boldsymbol{A}^{\forall\exists}$ is not AE regular, but $\boldsymbol{A}^{\forall\exists}x = \boldsymbol{b}^{\forall\exists}$ is AE solvable.

AE Regularity

AE Regularity

- is hard to check (generalizes regularity of $\boldsymbol{A}^{\forall}$)
- even regularity of $\boldsymbol{A}^{\exists}$ is open problem

Theorem

$\boldsymbol{A}^{\exists}$ is regular iff some vertex matrix ($a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$) is nonsingular.

Open Problems

- Is there a simpler characterization?
- What is the computational complexity?

Conjecture

$\boldsymbol{A}^{\exists}$ is regular iff it has no submatrix of size $k \times \ell$ that is real and has the rank $k + \ell - n - 1$.

Interval M-matrix

$A \in \mathbb{R}^{n\times n}$ is an M-matrix if $a_{ij} \le 0$ for $i \ne j$ and $A^{-1} \ge 0$.

Theorem ($\forall$ case, Barth & Nuding, 1974)

$\boldsymbol{A}^{\forall} \in \mathbb{IR}^{n\times n}$ is an M-matrix iff $\underline{A}$ is an M-matrix and $\overline{A}_{ij} \le 0$ for $i \ne j$.

Theorem ($\exists$ case)

Let $\boldsymbol{A}^{\exists} \in \mathbb{IR}^{n\times n}$ and define $\tilde{A} \in \boldsymbol{A}^{\exists}$ as follows

$$\tilde{a}_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } i = j, \\ \arg\min\{|a_{ij}| : a_{ij} \in \boldsymbol{a}^{\exists}_{ij}\} & \text{if } i \ne j. \end{cases} \qquad (*)$$

Then $\boldsymbol{A}^{\exists}$ is an M-matrix iff $\tilde{A}$ is an M-matrix.

Theorem ($\forall\exists$ case)

Denote by $\tilde{A} = \underline{A}^{\forall} + \tilde{A}^{\exists}$ the matrix from $(*)$ corresponding to $\underline{A}^{\forall} + \boldsymbol{A}^{\exists}$. Then $\boldsymbol{A}^{\forall\exists}$ is AE M-matrix iff $\tilde{A}$ is M-matrix and $(\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} \le 0$ for $i \ne j$.

Interval H-matrix

$A \in \mathbb{R}^{n \times n}$ is called an H-matrix, if the comparison matrix $\langle A \rangle$ is an M-matrix, where $\langle A \rangle_{ii} = |a_{ii}|$ and $\langle A \rangle_{ij} = -|a_{ij}|$ for $i \neq j$.

Magnitude and Mignitude

$$\begin{aligned} \operatorname{mag}(\boldsymbol{a}) &= \max\{|a| : a \in \boldsymbol{a}\} = |a_c| + a_\Delta, \\ \operatorname{mig}(\boldsymbol{a}) &= \min\{|a| : a \in \boldsymbol{a}\} = \begin{cases} 0 & \text{if } 0 \in \boldsymbol{a}, \\ \min(|\underline{a}|, |\overline{a}|) & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem (∀ case, Neumaier 1984)

Let $\boldsymbol{A}^\forall \in \mathbb{IR}^{n \times n}$ and define $\tilde{A} \in \boldsymbol{A}^\forall$ as follows

$$\tilde{a}_{ij} = \begin{cases} \operatorname{mig}(\boldsymbol{a}_{ij}^\forall) & \text{if } i = j, \\ -\operatorname{mag}(\boldsymbol{a}_{ij}^\forall) & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}^\forall$ is an H-matrix iff $\tilde{A}$ is an M-matrix.

Interval H-matrix

Theorem ($\exists$ case)

Let $\boldsymbol{A}^{\exists} \in \mathbb{IR}^{n\times n}$ *and define* $\tilde{A} \in \boldsymbol{A}^{\exists}$ *as follows*

$$\tilde{a}_{ij} = \begin{cases} \mathrm{mag}(\boldsymbol{a}^{\exists}_{ij}) & \text{if } i = j, \\ -\mathrm{mig}(\boldsymbol{a}^{\exists}_{ij}) & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}^{\exists}$ *is an H-matrix iff* $\tilde{A}$ *is an M-matrix.*

Theorem ($\forall\exists$ case)

Define the matrix $\tilde{A}$ *as follows*

$$\tilde{a}^{\exists}_{ij} = \begin{cases} \mathrm{mig}(\boldsymbol{a}^{\forall}_{ij}) + \mathrm{mag}(\boldsymbol{a}^{\exists}_{ij}) & \text{if } i = j, \\ -\mathrm{mag}(\boldsymbol{a}^{\forall}_{ij}) - \mathrm{mig}(\boldsymbol{a}^{\exists}_{ij}) & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}^{\forall\exists}$ *is an AE H-matrix if and only if* $\tilde{A}$ *is an M-matrix.*

Interval H-matrix

Theorem

If $A_c^{\forall}$ is an M-matrix, then $\boldsymbol{A}^{\forall}$ is regular iff $\boldsymbol{A}^{\forall}$ is H-matrix.

Example

For AE quantification, this statement is no longer valid.

$$\boldsymbol{A}^{\forall\exists} = \begin{pmatrix} [0.8, 1]^{\exists} & -[0, 1]^{\forall} \\ -1 & 1 \end{pmatrix}.$$

Then $A_c^{\forall\exists}$ is an M-matrix and $\boldsymbol{A}^{\forall\exists}$ is AE regular, but not an AE H-matrix.

Structured Quantifiers Position

Theorem

The square interval matrix

$$\boldsymbol{A}^{\forall\exists} = \begin{pmatrix} \boldsymbol{B}^{\forall} & \boldsymbol{b}^{\exists} \\ \boldsymbol{C}^{\forall} & \boldsymbol{c}^{\forall} \end{pmatrix}, \quad \text{where} \quad b^{\exists}_{\Delta} > 0,$$

is AE regular iff $(\boldsymbol{B}^T \ \boldsymbol{C}^T)^{\forall}$ *and* $(\boldsymbol{C} \ \boldsymbol{c})^{\forall}$ *have full row rank.*

Theorem

Let $\boldsymbol{B} \in \mathbb{IR}^{n\times k}$ *and* $\boldsymbol{C} \in \mathbb{IR}^{n\times(n-k)}$ *with* $C_{\Delta} > 0$. *Then* $(\boldsymbol{B}^{\forall} \ \boldsymbol{C}^{\exists})$ *is AE regular iff* $\boldsymbol{B}^{\forall}$ *has full column rank.*

Conjecture

The square interval matrix

$$\begin{pmatrix} \boldsymbol{B}^{\forall} & \boldsymbol{D}^{\exists} \\ \boldsymbol{C}^{\forall} & \boldsymbol{E}^{\forall} \end{pmatrix}, \quad \text{where} \quad D^{\exists}_{\Delta} > 0,$$

is AE regular iff $(\boldsymbol{B}^T \ \boldsymbol{C}^T)^{\forall}$ *and* $(\boldsymbol{C} \ \boldsymbol{E})^{\forall}$ *have full row rank.*

Conclusion

Summary

- We introduced a generalized regularity concept of interval matrices.
- Easy to handle cases:
  - M-matrices
  - H-matrices
  - special position of quantifiers

Many Open Problems

- AE regularity of $\boldsymbol{A}^{\forall\exists}$: characterization, ...
- regularity of $\boldsymbol{A}^{\exists}$: complexity, characterization, ...
- structured quantifiers position conjecture
- identifying other polynomial cases