Generalized Regularity and Singularity of Interval Matrices

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Interval Matrices – Motivation

Where interval data do appear

- numerical analysis (handling rounding errors)
  \[ \frac{1}{3} \in [0.33333333333333, 0.33333333333334] \]
  \[ \pi \in [3.1415926535897932384, 3.1415926535897932385] \]

- constraint solving and global optimization
  - find robot singularities, where it may breakdown
  - check joint angles \([0, 180]^\circ\)
  - find minimum of \(f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))\)

- modelling uncertainty
  - temperature measured in Da Nang \(29^\circ C \pm 1^\circ C\)
Interval Matrix and Regularity

**Definition**

**Interval Matrix**
An interval matrix is the family of matrices

\[ \mathcal{A} = \{ A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \bar{A} \}, \]

The midpoint and the radius matrices are defined as

\[ A_c := \frac{1}{2}(A + \bar{A}), \quad A_\Delta := \frac{1}{2}(\bar{A} - A). \]

The set of all interval \( m \times n \) matrices is denoted by \( \mathbb{I} \mathbb{R}^{m \times n} \).

**Regularity**

\( A \in \mathbb{I} \mathbb{R}^{n \times n} \) is regular if each \( A \in \mathcal{A} \) is nonsingular.
If $A \in \mathbb{IR}^{n \times n}$ is regular, then each realization of $Ax = b$ has a unique solution and the solution set

$$\{x \in \mathbb{R}^n : \exists A \in A \exists b \in b : Ax = b\}$$

is a nonempty and bounded (nonconvex) polyhedron.

Checking regularity is a co-NP-hard problem (Poljak & Rohn, 1988).
Disjointly split $A \in \mathbb{IR}^{n \times n}$ as

$$A = A^\forall + A^\exists,$$

where
- $A^\forall$ is the interval matrix comprising universally quantified coefficients,
- $A^\exists$ concerns existentially quantified coefficients.

Example

$$A = \begin{pmatrix} [-1, 1]^\forall & [1, 2]^\exists \\ 23 & [5, 6]^\exists \end{pmatrix}$$

Definition

An interval matrix $A = A^\forall + A^\exists$ is called AE regular if

$$\forall A^\forall \in A^\forall \exists A^\exists \in A^\exists$$ such that $A = A^\forall + A^\exists$ is nonsingular.
Motivation I.

Quantified Interval Linear Programming

\[ \min \ c^T x \ \text{subject to} \ A x = b, \ x \geq 0 \]

Traditional Problems

- range of optimal values
- basis stability
- optimal solution set
- robust optimization

AE quantification

- intensively studied with \( \forall \exists \) quantification now,
- robust optimization,
- need for \( \forall \exists \) quantification \( (\sum i w_i x_i = 1) \).
Motivation II.

Definition (AE Solvability of Interval Linear Systems)

\( A^\forall x = b^\forall \) is AE solvable if

\[
\forall A^\forall \in A^\forall, \forall b^\forall \in b^\forall, \exists A^\exists \in A^\exists, \exists b^\exists \in b^\exists : (A^\forall + A^\exists)x = b^\forall + b^\exists \text{ is solvable}.
\]

Theorem

If \( A^\forall \) is AE regular, then \( A^\forall x = b^\forall \) is AE solvable for each \( b^\forall \).

The converse is not true in general.

Example

The counter-example for the converse direction is

\[
A^\forall = 0, \quad A^\exists = \begin{pmatrix} 0 & [-1, 1] \\ 0 & [-1, 1] \end{pmatrix}.
\]

Then \( A^\forall \) is not AE regular, but \( A^\forall x = b^\forall \) is AE solvable.
AE Regularity

- is hard to check (generalizes regularity of $A^{\forall}$)
- even regularity of $A^\exists$ is open problem

Theorem

$A^\exists$ is regular iff some vertex matrix ($a_{ij} \in \{a_{ij}, \bar{a}_{ij}\}$) is nonsingular.

Open Problems

- Is there a simpler characterization?
- What is the computational complexity?

Conjecture

$A^\exists$ is regular iff it has no submatrix of size $k \times \ell$ that is real and has the rank $k + \ell - n - 1$. 
Interval M-matrix

$A \in \mathbb{R}^{n \times n}$ is an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$.

**Theorem (\forall case, Barth & Nuding, 1974)**

$A^{\forall} \in \mathbb{IR}^{n \times n}$ is an M-matrix iff $A$ is an M-matrix and $\overline{A}_{ij} \leq 0$ for $i \neq j$.

**Theorem (\exists case)**

Let $A^{\exists} \in \mathbb{IR}^{n \times n}$ and define $\tilde{A} \in A^{\exists}$ as follows

$$\tilde{a}_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } i = j, \\ \arg \min \{|a_{ij}| : a_{ij} \in a^{\exists}_{ij}\} & \text{if } i \neq j. \end{cases} \quad (*)$$

Then $A^{\exists}$ is an M-matrix iff $\tilde{A}$ is an M-matrix.

**Theorem (\forall\exists case)**

Denote by $\tilde{A} = A^{\forall} + \tilde{A}^{\exists}$ the matrix from (*) corresponding to $A^{\forall} + A^{\exists}$.

Then $A^{\forall\exists}$ is AE M-matrix iff $\tilde{A}$ is M-matrix and $(\overline{A}^{\forall} + \tilde{A}^{\exists})_{ij} \leq 0$ for $i \neq j$. 
Interval H-matrix

\( A \in \mathbb{R}^{n \times n} \) is called an H-matrix, if the comparison matrix \( \langle A \rangle \) is an M-matrix, where \( \langle A \rangle_{ii} = |a_{ii}| \) and \( \langle A \rangle_{ij} = -|a_{ij}| \) for \( i \neq j \).

Magnitude and Mignitude

\[
\text{mag}(a) = \max\{|a| : a \in a\} = |a_c| + a_\Delta,
\]

\[
\text{mig}(a) = \min\{|a| : a \in a\} = \begin{cases} 0 & \text{if } 0 \in a, \\ \min(|a|, |\overline{a}|) & \text{otherwise}. \end{cases}
\]

Theorem (\( \forall \) case, Neumaier 1984)

Let \( A^\forall \in \mathbb{IR}^{n \times n} \) and define \( \tilde{A} \in A^\forall \) as follows

\[
\tilde{a}_{ij} = \begin{cases} \text{mig}(a^\forall_{ij}) & \text{if } i = j, \\ -\text{mag}(a^\forall_{ij}) & \text{if } i \neq j. \end{cases}
\]

Then \( A^\forall \) is an H-matrix iff \( \tilde{A} \) is an M-matrix.
**Interval H-matrix**

### Theorem (∃ case)

Let $A^\exists \in \mathbb{IR}^{n \times n}$ and define $\tilde{A} \in A^\exists$ as follows

$$\tilde{a}_{ij} = \begin{cases} 
\text{mag}(a^\exists_{ij}) & \text{if } i = j, \\
-\text{mig}(a^\exists_{ij}) & \text{if } i \neq j.
\end{cases}$$

Then $A^\exists$ is an H-matrix iff $\tilde{A}$ is an M-matrix.

### Theorem (∀∃ case)

Define the matrix $\tilde{A}$ as follows

$$\tilde{a}^\exists_{ij} = \begin{cases} 
\text{mig}(a^\forall_{ij}) + \text{mag}(a^\exists_{ij}) & \text{if } i = j, \\
-\text{mag}(a^\forall_{ij}) - \text{mig}(a^\exists_{ij}) & \text{if } i \neq j.
\end{cases}$$

Then $A^{\forall\exists}$ is an AE H-matrix if and only if $\tilde{A}$ is an M-matrix.
**Theorem**

If $A_c^\forall$ is an M-matrix, then $A^\forall$ is regular iff $A^\forall$ is H-matrix.

**Example**

For AE quantification, this statement is no longer valid.

$$A^{\forall\exists} = \begin{pmatrix} [0.8, 1]^\exists & -[0, 1]^\forall \\ -1 & 1 \end{pmatrix}.$$  

Then $A_c^{\forall\exists}$ is an M-matrix and $A^{\forall\exists}$ is AE regular, but not an AE H-matrix.
The square interval matrix

\[ A^\exists = \begin{pmatrix} B^\forall & b^\exists \\ C^\forall & c^\forall \end{pmatrix}, \text{ where } b_{\Delta}^\exists > 0, \]

is AE regular iff \((B^T C^T)^\forall\) and \((C c)^\forall\) have full row rank.

Let \(B \in \mathbb{IR}^{n \times k}\) and \(C \in \mathbb{IR}^{n \times (n-k)}\) with \(C_\Delta > 0\). Then \((B^\forall C^\exists)\) is AE regular iff \(B^\forall\) has full column rank.

The square interval matrix

\[
\begin{pmatrix} B^\forall & D^\exists \\ C^\forall & E^\forall \end{pmatrix}, \text{ where } D_{\Delta}^\exists > 0,
\]

is AE regular iff \((B^T C^T)^\forall\) and \((C E)^\forall\) have full row rank.
We introduced a generalized regularity concept of interval matrices.

Easy to handle cases:
- M-matrices
- H-matrices
- special position of quantifiers

Many Open Problems
- AE regularity of $A^{\forall\exists}$: characterization, ...
- regularity of $A^{\exists}$: complexity, characterization, ...
- structured quantifiers position conjecture
- identifying other polynomial cases