Interval Linear Programming

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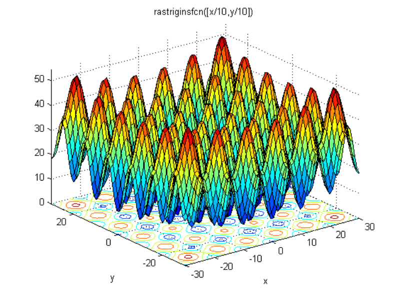
Outline

Next Section

Interval Data – Motivation

Where interval data do appear

- numerical analysis (handling rounding errors)
  - $\frac{1}{3} \in [0.33333333333333, 0.33333333333334]$
  - $\pi \in [3.1415926535897932384, 3.1415926535897932385]$.
- constraint solving and global optimization
  - find robot singularities, where it may breakdown
    check joint angles $[0, 180]^\circ$
  - find minimum of $f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))$



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- statistical estimation
  - confidence intervals, prediction intervals (future prices,...)
- measurement errors
  - fuel consumption, stiffness in truss construction, velocity ($75 \pm 2\,\mathrm{km/h}$)
- discretization
  - time is split in days
  - day range of stock prices – daily min / max
- missing data

Interval Matrices

Definition (Interval matrix)

An interval matrix is the family of matrices

$$\boldsymbol{A} = \{A \in \mathbb{R}^{m\times n} : \underline{A} \le A \le \overline{A}\},$$

The midpoint and the radius matrices are defined as

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all interval $m \times n$ matrices is denoted by $\mathbb{IR}^{m\times n}$.

Important notice

We consider intervals in a set sense, no distribution, no fuzzy shape.

Introduction

Linear programming – three basic forms

$$f(A,b,c) \equiv \min\ c^T x \ \text{ subject to }\ Ax = b,\ x \geq 0,$$
$$f(A,b,c) \equiv \min\ c^T x \ \text{ subject to }\ Ax \leq b,$$
$$f(A,b,c) \equiv \min\ c^T x \ \text{ subject to }\ Ax \leq b,\ x \geq 0.$$

Interval linear programming

Family of linear programs with $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$, in short

$$f(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) \equiv \min \boldsymbol{c}^T x \ \text{ subject to }\ \boldsymbol{A}x \overset{(\leq)}{=} \boldsymbol{b},\ (x \geq 0).$$

The three forms are not transformable between each other!

Main goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

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Optimal Value Range

Definition

$$\underline{f} := \min \; f(A,b,c) \;\; \text{subject to} \;\; A \in \boldsymbol{A},\; b \in \boldsymbol{b},\; c \in \boldsymbol{c},$$
$$\overline{f} := \max \; f(A,b,c) \;\; \text{subject to} \;\; A \in \boldsymbol{A},\; b \in \boldsymbol{b},\; c \in \boldsymbol{c}.$$

Observation

If $f(A,b,c)$ is continuous on $\boldsymbol{A} \times \boldsymbol{b} \times \boldsymbol{c}$, then $\underline{f}$ and $\overline{f}$ are finite and $f(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) = [\underline{f}, \overline{f}]$.

Example (Bereanu, 1978)

$$\max \; x_1 \;\; \text{subject to} \;\; x_1 \le [1,2],\; [-1,1]x_1 \le 0,\; -x_1 \le 0.$$

The image of the optimal value is $\{0\} \cup [1,2]$.

Open problems

How many components of $f(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$? Always closed?

Optimal Value Range

Theorem (Wets, 1985, Mostafaee et al., 2016)

Suppose that both interval linear systems

$$\boldsymbol{A}x = 0,\ x \geq 0,\ \boldsymbol{c}^T x \leq 0$$

and

$$\boldsymbol{A}^T y \leq 0,\ \boldsymbol{b}^T y \geq 0$$

have only trivial solution. Then $f(A, b, c)$ *is continuous on* $\boldsymbol{A} \times \boldsymbol{b} \times \boldsymbol{c}$.

Theorem

It is NP-hard to check if the value f *is attained for a given* $f \in [\underline{f}, \overline{f}]$.

Optimal Value Range

Theorem (Vajda, 1961)

We have for type $(\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0)$

$$\underline{f} = \min \underline{c}^T x \ \ \textit{subject to} \ \ \underline{A}x \leq \overline{b},\ x \geq 0,$$
$$\overline{f} = \min \overline{c}^T x \ \ \textit{subject to} \ \ \overline{A}x \leq \underline{b},\ x \geq 0.$$

Theorem (Machost, 1970, Rohn, 1984)

We have for type $(\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0)$

$$\underline{f} = \min \underline{c}^T x \ \ \textit{subject to} \ \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0,$$
$$\overline{f} = \max_{s \in \{\pm 1\}^m} f(A_c - \operatorname{diag}(s) A_\Delta, b_c + \operatorname{diag}(s) b_\Delta, \overline{c}).$$

Theorem (Rohn (1997), Gabrel et al. (2008))

- *checking* $\overline{f} = \infty$ *is NP-hard*
- *checking* $\overline{f} \geq 1$ *is strongly NP-hard (with* A, c *crisp and* $\overline{f} < \infty$*)*

Optimal Value Range

Algorithm (Optimal value range $[\underline{f}, \overline{f}]$)

1. Compute

$$\underline{f} := \inf \ c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ x \in \mathcal{M},$$

   where $\mathcal{M}$ is the primal solution set.
2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute

$$\overline{\varphi} := \sup \ b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ y \in \mathcal{N},$$

   where $\mathcal{N}$ is the dual solution set.
4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If the primal problem is strongly feasible, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

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Optimal Solution Set

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \;\; \text{subject to} \;\; Ax = b, \; x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, and $c \in \boldsymbol{c}$ such that

$$Ax = b, \; x \geq 0, \; A^T y \leq c, \; c^T x = b^T y,$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

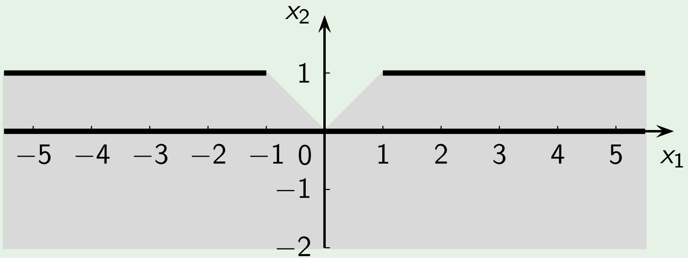
Optimal Solution Set

Example (Garajová, 2016)

The optimal solution set may be disconnected and nonconvex.

Consider the interval LP problem

$$\max \ x_2 \ \text{ subject to } \ [-1,1]x_1 + x_2 \le 0, \ x_2 \le 1.$$



Optimal Solution Set

Theorem (Garajová, H., 2016)

The set of optimal solutions $\mathcal{S}$ of the interval linear program (with real A)

$$\min \ \boldsymbol{c}^T x \ \ \textit{subject to} \ \ Ax = \boldsymbol{b}, \ x \geq 0$$

is a path-connected union of at most 2^n convex polyhedra.

Observation

If b is real in addition, then $\mathcal{S}$ is formed by a union of some faces of the feasible set.

Open Problems

- More about topology of the optimal solution set $\mathcal{S}$ (Is it always polyhedral?),
- characterization of $\mathcal{S}$,
- tight approximation of $\mathcal{S}$.

Next Section

Basis Stability

Definition

The interval linear programming problem

$$\min \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x = \boldsymbol{b},\ x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\underline{f} = \min\ \underline{c}_B^T x \ \textit{ subject to } \ \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$
$$\overline{f} = \max\ \overline{c}_B^T x \ \textit{ subject to } \ \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Moreover, $f(A, b, c)$ is continuous and $f(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = [\underline{f}, \overline{f}]$.

Under the unique B-stability, the set of all optimal solutions reads

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

(Otherwise each realization has at least one optimal solution in this set.)

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C1

- C1 says that $\boldsymbol{A}_B$ is regular;
- co-NP-hard problem;
- Beeck's sufficient condition: $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$.

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C2

- C2 says that the solution set to $\boldsymbol{A}_B x_B = \boldsymbol{b}$ lies in $\mathbb{R}^n_+$;
- sufficient condition: check of some enclosure to $\boldsymbol{A}_B x_B = \boldsymbol{b}$.

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C3

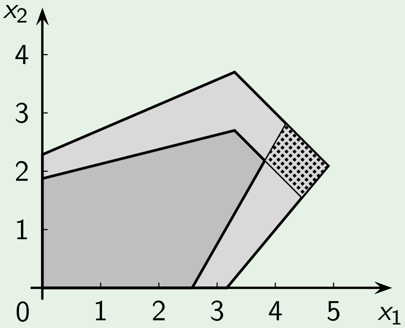
- C2 says that $\boldsymbol{A}_N^T y \leq \boldsymbol{c}_N,\ \boldsymbol{A}_B^T y = \boldsymbol{c}_B$ is strongly feasible;
- co-NP-hard problem;
- sufficient condition:
  $(\boldsymbol{A}_N^T)\boldsymbol{y} \leq \underline{c}_N$, where $\boldsymbol{y}$ is an enclosure to $\boldsymbol{A}_B^T y = \boldsymbol{c}_B$.

Basis Stability – Example

Example

Consider an interval linear program

$$\max\ ([5,6],[1,2])^T x \ \text{ s.t. } \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix},\ x \geq 0.$$



- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area

Basis Stability – Interval Right-Hand Side

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$ for each $b \in \boldsymbol{b}$.

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Condition C1

- C1 and C3 are trivial
- C2 is simplified to

$$\underline{A_B^{-1}\boldsymbol{b}} \geq 0,$$

  which is easily verified by interval arithmetic
- overall complexity: polynomial

Basis Stability – Interval Objective Function

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$ for each $c \in \boldsymbol{c}$

Condition C1

- C1 and C2 are trivial
- C3 is simplified to

$$A_N^T y \leq \boldsymbol{c}_N,\ A_B^T y = \boldsymbol{c}_B$$

or,

$$\overline{(A_N^T A_B^{-T})\boldsymbol{c}_B} \leq \underline{c}_N.$$

- overall complexity: polynomial

Next Section

Applications

Real-life applications

- Transportation problems with uncertain demands, suppliers, and/or costs.
- Networks flows with uncertain capacities.
- Diet problems with uncertain amounts of nutrients in foods.
- Portfolio selection with uncertain rewards.
- Matrix games with uncertain payoffs.

Technical applications

- Tool for global optimization.
- Measure of sensitivity of linear programs.

Verification

- Handle rigorously numerics of real-valued linear programs.

Verification – Motivation

Example (Rump, 1988)

Consider the expression

$$f = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b},$$

with

$$a = 77617, \quad b = 33096.$$

Calculations from 80s gave

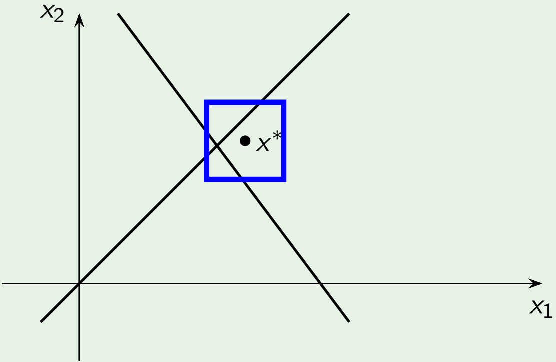
single precision	$f \approx 1.172603\ldots$
double precision	$f \approx 1.1726039400531\ldots$
extended precision	$f \approx 1.172603940053178\ldots$
the **true** value	$f = -0.827386\ldots$

Verification

Verification of a system of linear equations

Given a real system $Ax = b$ and x^* approximate solution, find $x^* \in \boldsymbol{x} \in \mathbb{IR}^n$ such that $A^{-1}b \in \boldsymbol{x}$.

Example



Verification in Linear Programming

Consider a linear program

$$\min\ c^T x \ \text{ subject to }\ Ax = b,\ x \geq 0.$$

Let B^* be an optimal basis, f^* optimal value and x^* optimal solution. All these are numerically computed.

Verification of the optimal basis (Jansson, 1988)

- confirmation that B^* is (unique) optimal basis,

Verification of the optimal value (Neumaier & Shcherbina, 2004)

- finding $f^* \in \boldsymbol{f} \in \mathbb{IR}$ such that $\boldsymbol{f}$ contains the optimal value,

Verification of the optimal solution

- finding $x^* \in \boldsymbol{x} \in \mathbb{IR}^n$ such that $\boldsymbol{x}$ contains the (unique) optimal solution.

Verification of Optimal Basis

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Verification of condition C2

- Compute verification interval $\boldsymbol{x}_B$ for $A_B x_B = b$,
- check $\underline{x}_B \geq 0$ (resp. $\underline{x}_B > 0$ for uniqueness)

Verification of condition C3

- Compute verification interval $\boldsymbol{y}$ for $A_B^T y = c_B$,
- check $c_N^T - \boldsymbol{y}^T A_N \geq 0$ (resp. $c_N^T - \boldsymbol{y}^T A_N > 0$ for uniqueness).

Conclusion

Conclusion

Interval linear programming provides techniques for

- studying effects of data variations on optimal value and optimal solutions
- processing state space of parameters
- calculating bounds
- handling numerical errors

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