A Novel Data Envelopment Analysis Ranking Based on a Robust Approach

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CCR model for DEA

CCR model for DEA

The classical CCR model (Charnes et al., 1978) for ranking the decision making unit DMU_0 can be formulated as a linear program

$$\max \; y_0^T u \;\; \text{subject to} \;\; x_0^T v \leq 1, \;\; Yu - Xv \leq 0, \;\; u, v \geq 0,$$

where

- $x_0 \in \mathbb{R}^{n_1}$ is the input nonnegative vector for DMU_0,
- $y_0 \in \mathbb{R}^{n_2}$ is the output nonnegative vector for DMU_0,
- $X \in \mathbb{R}^{m \times n_1}$ is the input nonnegative matrix for the other DMU's, in particular, the ith row of X is the input vector for the ith DMU,
- $Y \in \mathbb{R}^{m \times n_2}$ is the output nonnegative matrix for the other DMU's, in particular, the ith row of Y is the output vector for the ith DMU,
- u and v are vectors of variables representing output and input weights, respectively.

Our goal

Discussion

Many other models exist.

Our goal

Introduce a new efficiency ranking based on a robustness point of view.

Why?

Besides the positive properties of the classical model:

- quantify stability/robustness
- measure distance to (in)efficiency
- handle precise as well as imprecise data

Other approaches

- robust models for imprecise data

Novel ranking – idea

The idea of novel robust ranking

- determine the largest allowable variations of the input and output data such that DMU_0 remains efficient (for efficient DMU's), or
- the smallest possible variation of the input and output data such that DMU_0 becomes efficient (for inefficient DMU's)
- the corresponding coefficient of variations gives us a new ranking

Novel ranking – formalization

δ-neighborhood

Define δ-neighborhood of the data as

$$\begin{aligned}\mathcal{O}_\delta(x_0, y_0, X, Y) = \{(x_0', y_0', X', Y') \colon & |x_{ij}' - x_{ij}| \le \delta x_{ij}, \\ & |y_{ik}' - y_{ik}| \le \delta y_{ik},\ \forall i, j, k\}.\end{aligned}$$

New ranking

If DMU_0 is efficient, then its ranking is defined as $r = 1 + \delta^*$, where

$$\begin{aligned}\delta^* = \max\{\delta \colon & \text{DMU}_0 \text{ is efficient for all} \\ & (x_0', y_0', X', Y') \in \mathcal{O}_\delta(x_0, y_0, X, Y)\}.\end{aligned}$$

If DMU_0 is inefficient, then its ranking is defined as $r = 1 + \delta^*$, where

$$\begin{aligned}\delta^* = -\min\{\delta \colon & \text{DMU}_0 \text{ is efficient for some} \\ & (x_0', y_0', X', Y') \in \mathcal{O}_\delta(x_0, y_0, X, Y)\}.\end{aligned}$$

Notice that the maximum (minimum) value needn't be attained.

Novel ranking – computation

Theorem

We have

$$\delta^* = \max\ \delta \quad \textit{subject to} \quad (1-\delta) y_0^T u \geq 1,\ (1+\delta) x_0^T v \leq 1,$$
$$(1+\delta) Yu - (1-\delta) Xv \leq 0,\ u, v \geq 0.$$

Properties

- it is a nonlinear programming problem in variables δ, u, v
- still efficiently computable
- in a class of generalized linear fractional programming problems, which have the form of

$$\min\ \lambda \quad \text{subject to} \quad Ax \leq \lambda Bx,\ Cx \leq c,\ x \geq 0,$$

where $Bx \geq 0$ holds for all x satisfying $Cx \leq c$, $x \geq 0$. Equivalently

$$\min\ \left(\max_i \frac{(Ax)_i}{(Bx)_i}\right) \quad \text{subject to} \quad Cx \leq c,\ x \geq 0.$$

Novel ranking – computation

The resulting linearized model

The ranking is $r = 1 + \delta^*$, where δ^* is computed by the linear program

$$\delta^* = \max\ \delta \ \text{ subject to } \ y_0^T \tilde{u} \geq 1 + \delta,\ x_0^T \tilde{v} \leq 1 - \delta,$$
$$Y\tilde{u} - X\tilde{v} \leq 0,\ \tilde{u}, \tilde{v} \geq 0.$$

Properties

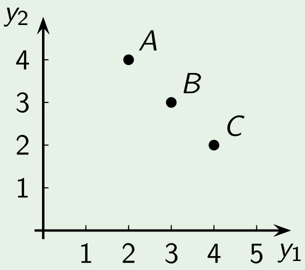
- invariant to scaling the units of input and output data
- $r \in [0, 2]$
- $r \geq 1$ if and only if DMU_0 is efficient
- $r < 1$ if and only if DMU_0 is inefficient
- does not change (in)efficiency of the classical ranking,
  even the order is not changed (for both linear and nonlinear models)

Novel ranking – properties

New ranking as a robustness measure

- we can use r as the measure of inefficiency and efficiency
- if $r = 1 + \delta^* \geq 1$, then DMU_0 is efficient for any variation of the data up to $100\delta^*\%$ of their nominal values; moreover, all data coefficients may vary simultaneously and independently to each other.

Example



Outputs of DMU's A, B and C are

$$A : (2, 4), \quad B : (3, 3), \quad C : (4, 2).$$

- the classical ranks: 1, 1, 1
- a slight change of the outputs of B makes it inefficient
- our ranks are 1.1429, 1, 1.1429
- B is borderline; A, C are stable (under 14.29% perturbation)

Novel ranking – properties

New ranking as a global ranking

- the novel ranking is naturally normalized
- use for comparing DMU's from different, even unrelated models

Example

Suppose that we have a ranking of banks like

$$1.0062,\ 0.986,\ 1.0397,\ 1.024,\ 0.97263,\ 1.0009,\ 1.0438,\ 0.96441,$$

and suppose that we have a ranking of hospitals like

$$1.21,\ 0.65338,\ 1.3254,\ 0.6799,\ 1.0382,\ 0.60379,\ 0.89957,\ 1.2454.$$

- In the first case, all the banks have very similar ranking, no substantial difference in their performance
- performance of particular hospitals differs a lot
- there are some considerably efficient and some highly inefficient
- this shows the universal feature of our approach to ranking

Handling interval data

- suppose we are given interval data $[\underline{x}_0, \overline{x}_0]$, $[\underline{X}, \overline{X}]$, $[\underline{Y}, \overline{Y}]$ and $[\underline{x}_0, \overline{x}_0]$
- the best case rank $\overline{r}$ happens in the setting

$$x_0 := \underline{x}_0, \; y_0 := \overline{y}_0, \; X := \overline{X} \text{ and } Y := \underline{Y}$$

- the worst case rank $\underline{r}$ happens for

$$x_0 := \overline{x}_0, \; y_0 := \underline{y}_0, \; X := \underline{X} \text{ and } Y := \overline{Y}$$

- by solving two real-valued problems we get the efficiency range $[\underline{r}, \overline{r}]$
- if $\underline{r} \geq 1$, then DMU_0 is always efficient
- if $\overline{r} < 1$, then DMU_0 is never efficient

Novel ranking – additional DMU / input or output

Additional DMU

- additional DMU $=$ a novel constraint in the optimization model
- the ranking cannot increase

Theorem

Suppose DMU_0 is efficient. The set of all values $x_a \in \mathbb{R}^{n_1}$ and $y_a \in \mathbb{R}^{n_1}$ causing DMU_0 to be inefficient forms an interior of a convex polyhedron.

Additional inputs or outputs

- the ranking cannot decrease

Theorem

Suppose DMU_0 is inefficient. The set of all values $(y_b, y_c) \in \mathbb{R}^{1+m}$ causing DMU_0 to be efficient forms an interior of a convex polyhedron.

Novel ranking – Example 1

Example (Cooper et al., 2007)

Ranking hospitals:

	INPUT		OUTPUT				
DMU	doctors	nurses	outpatients	inpatients	classical eff.	new eff. (lin.)	new eff.
A	20	151	100	90	1	1.1696	1.1708
B	19	131	150	50	1	1.0843	1.0845
C	25	160	160	55	0.8827	0.9377	0.9376
D	27	168	180	72	1	1.0079	1.0079
E	22	158	94	66	0.7635	0.8659	0.8653
F	55	255	230	90	0.8348	0.9100	0.9097
G	33	235	220	88	0.9020	0.9485	0.9484
H	31	206	152	80	0.7963	0.8866	0.8863
I	30	244	190	100	0.9604	0.9798	0.9798
J	50	268	250	100	0.8707	0.9309	0.9307
K	53	306	260	147	0.9551	0.9770	0.9770
L	38	284	250	120	0.9582	0.9787	0.9787

Novel ranking – Example 2

Example (Entani et al., 2002, He et al., 2016, . . .)

DMU	X_1	Y_1	Y_2	efficiency from Refs	novel efficiency
A	1	[0.8, 1.2]	[7.50, 8.50]	[1, 1]	[1.0169, 1.1148]
B	1	[1.8, 2.2]	[2.50, 3.50]	[0.4222, 0.6227]	[0.5937, 0.7675]
C	1	[1.6, 2.4]	[5.75, 6.25]	[0.7297, 0.9167]	[0.8437, 0.9566]
D	1	[2.5, 3.5]	[2.75, 3.25]	[0.5247, 0.7809]	[0.6882, 0.8770]
E	1	[2.8, 3.2]	[6.75, 7.25]	[0.9646, 1]	[0.9819, 1.1292]
F	1	[3.8, 4.2]	[1.83, 2.17]	[0.6131, 0.7806]	[0.7601, 0.8768]
G	1	[3.4, 4.6]	[4.50, 5.50]	[0.7940, 1]	[0.8852, 1.0643]
H	1	[4.7, 5.3]	[1.50, 2.50]	[0.6984, 0.9635]	[0.8224, 0.9814]
I	1	[5.6, 6.4]	[1.67, 2.33]	[0.8229, 1]	[0.9028, 1.0482]
J	1	[6.7, 7.3]	[0.75, 1.25]	[1, 1]	[1.0229, 1.1318]

- A and J are efficient for each realization
- B, C, D, F, and H are inefficient for each realization
  B or F are far to efficiency while H is possibly closer
- E is either efficient or very close to efficiency for each realization.

Conclusion

Summary

New DEA ranking based on robustness of DMU's of their (in)efficiency with many attractive properties:

- efficiently computable by linear programming
- invariant with respect to scaling
- it gives a measure of efficiency as a distance to inefficiency and vice versa
- suitable as a universal ranking technique of DMU's of different models.
- suitable for further generalization – models with interval data etc.