# When dependencies do not matter?

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# Dependencies

## Example (Well Known)

What is the range of $f(x) = x^2 - x$ on $\boldsymbol{x} = [-1, 2]$?
Due to dependencies, different expressions lead to different enclosures:

$$\begin{aligned} \boldsymbol{x}^2 - \boldsymbol{x} &= [-2, 5], \\ \boldsymbol{x}(\boldsymbol{x} - 1) &= [-4, 2], \\ (\boldsymbol{x} - \tfrac{1}{2})^2 - \tfrac{1}{4} &= [-\tfrac{1}{4}, 2]. \end{aligned}$$

## Example

What is the range of $f(x) = x^2 - x$ on $\boldsymbol{x} = [1, 2]$? Now:

$$\begin{aligned} \boldsymbol{x}^2 - \boldsymbol{x} &= [-1, 3], \\ \boldsymbol{x}(\boldsymbol{x} - 1) &= [0, 2], \\ (\boldsymbol{x} - \tfrac{1}{2})^2 - \tfrac{1}{4} &= [0, 2]. \end{aligned}$$

Despite dependencies, the second evaluation is exact. (See Hansen, 1997)

# Another motivation

**Theorem (Rohn, 1994)**

$\boldsymbol{A}$ *(with* $\overline{A} = \overline{A}^T$, $\underline{A} = \underline{A}^T$ *) is positive (semi-)definite if and only if* $\boldsymbol{A}^S$ *is.*

**Theorem (Rohn, 2014)**

$\boldsymbol{A}x = \boldsymbol{b}$ *is strongly*[1] *solvable if and only if* $(D_s\boldsymbol{A})x \leq D_s\boldsymbol{b}$ *is strongly solvable for each* $s \in \{\pm 1\}^m$.

**Theorem (Li, 2015, and others before)**

$\boldsymbol{A}x = \boldsymbol{b}$ *is weakly*[2] *solvable if and only if* $\boldsymbol{A}x \leq \boldsymbol{b}$, $-\boldsymbol{A}x \leq -\boldsymbol{b}$ *is weakly solvable.*

[1] each realization is solvable
[2] some realization is solvable

# Problem Statement

**Questions**

- Any other examples when dependencies do not matter?
- What are the equivalent transformations?

# Weak Solvability of Standard Interval Linear Systems

**Theorem**

*$\boldsymbol{A}x = \boldsymbol{b}$ is weakly solvable if and only if $(\boldsymbol{A}D_s)x = \boldsymbol{b}$, $x \geq 0$ is weakly solvable for some $s \in \{\pm 1\}^n$.*

**Theorem**

*A system $\boldsymbol{A}x \leq \boldsymbol{b}$ is weakly solvable if and only if $(\boldsymbol{A}D_s)x \leq \boldsymbol{b}$, $x \geq 0$ is weakly solvable for some $s \in \{\pm 1\}^n$.*

# Strong Solvability of Standard Interval Linear Systems

**Theorem**

*$\boldsymbol{A}x = \boldsymbol{b}$ is strongly solvable if and only if the following is strongly solvable*

$$\boldsymbol{A}x^1 - \boldsymbol{A}x^2 = \boldsymbol{b}, \quad x^1, x^2 \geq 0$$

**Theorem**

*$\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly solvable if and only if the following is strongly solvable*

$$\boldsymbol{A}x^1 - \boldsymbol{A}x^2 \leq \boldsymbol{b}, \quad x^1, x^2 \geq 0$$

**Example**

By Rohn (2006), $\boldsymbol{A}x = \boldsymbol{b}$, $x \geq 0$ is strongly solvable if and only if

$$(A_c + D_s A_\Delta)x = b_c - D_s b_\Delta, \quad x \geq 0$$

is solvable $\forall s \in \{\pm 1\}^m$. Next, $\boldsymbol{A}x = \boldsymbol{b}$ is strongly solvable if and only if

$$(A_c + D_s A_\Delta)x^1 - (A_c - D_s A_\Delta)x^2 = (b_c - D_s b_\Delta), \quad x^1, x^2 \geq 0$$

is solvable $\forall s \in \{\pm 1\}^m$. This is a consequence of the above theorem!

# AE Solutions and Solvability

Split

$$\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}, \quad \boldsymbol{b} = \boldsymbol{b}^{\forall} + \boldsymbol{b}^{\exists},$$

where $\boldsymbol{A}^{\forall}$, $\boldsymbol{b}^{\forall}$ contain the universally quantified coefficients and $\boldsymbol{A}^{\exists}$, $\boldsymbol{b}^{\exists}$ the existential ones.

### Definition

$x \in \mathbb{R}^n$ is an *AE solution* of $\boldsymbol{A}x = \boldsymbol{b}$ if

$$\forall A^{\forall} \in \boldsymbol{A}^{\forall}\, \forall b^{\forall} \in \boldsymbol{b}^{\forall}\, \exists A^{\exists} \in \boldsymbol{A}^{\exists}\, \exists b^{\exists} \in \boldsymbol{b}^{\exists} : (A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists}.$$

### Definition

The interval system $\boldsymbol{A}x = \boldsymbol{b}$ is *AE solvable* if

$\forall A^{\forall} \in \boldsymbol{A}^{\forall}\, \forall b^{\forall} \in \boldsymbol{b}^{\forall}\, \exists A^{\exists} \in \boldsymbol{A}^{\exists}\, \exists b^{\exists} \in \boldsymbol{b}^{\exists} : (A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists}$ is solvable.

Analogously for inequalities.

# AE Solutions

**Theorem**

*The system* $\boldsymbol{A}x = \boldsymbol{b}$ *has an AE solution if and only if the system*

$$(\boldsymbol{A}D_s)x = \boldsymbol{b}, \quad x \geq 0$$

*has an AE solution for some* $s \in \{\pm 1\}^n$.

**Theorem**

*The system* $\boldsymbol{A}x \leq \boldsymbol{b}$ *has an AE solution if and only if the system*

$$(\boldsymbol{A}D_s)x \leq \boldsymbol{b}, \quad x \geq 0$$

*has an AE solution for some* $s \in \{\pm 1\}^n$.

**Theorem**

*The system* $\boldsymbol{A}x = \boldsymbol{b}$ *has an AE solution if and only if the system*

$$\boldsymbol{A}x \leq \boldsymbol{b}, \quad -\boldsymbol{A}x \leq -\boldsymbol{b},$$

*has an AE solution. Transf. to* $\boldsymbol{A}x^1 - \boldsymbol{A}x^2 = \boldsymbol{b}, \quad x^1, x^2 \geq 0$ *is not possible.*

# AE Solvability

**Theorem**

*Suppose that* ${A^{\exists}}_{\Delta} = 0$. *Then* $\boldsymbol{A}x = \boldsymbol{b}$ *is AE solvable if and only if*

$$\boldsymbol{A}x^1 - \boldsymbol{A}x^2 = \boldsymbol{b}, \quad x^1, x^2 \geq 0$$

*is AE solvable.*

**Theorem**

*Suppose that* ${A^{\exists}}_{\Delta} = 0$. *Then* $\boldsymbol{A}x \leq \boldsymbol{b}$ *is AE solvable if and only if*

$$\boldsymbol{A}x^1 - \boldsymbol{A}x^2 \leq \boldsymbol{b}, \quad x^1, x^2 \geq 0$$

*is AE solvable.*

**Theorem**

*The system* $\boldsymbol{A}x = \boldsymbol{b},\ \boldsymbol{C}x \leq \boldsymbol{d}$ *is AE solvable if and only if the system*

$$y = \boldsymbol{A}^{\forall}x - \boldsymbol{b}^{\forall}, \quad y = \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists}x, \quad z \geq \boldsymbol{C}^{\forall}x - \boldsymbol{d}^{\forall}, \quad z \leq \boldsymbol{d}^{\exists} - \boldsymbol{C}^{\exists}x$$

*is AE solvable.*

# Parametric Systems

Let $A(p), b(p)$ be functions of $p \in \boldsymbol{p} \in \mathbb{IR}^K$.

### Theorem

*The interval system*

$$A(p)x = b(p), \quad p \in \boldsymbol{p}$$

*is weakly solvable if and only if the interval system*

$$(A(p)D_s)x = b(p),\ x \geq 0, \quad p \in \boldsymbol{p}$$

*is weakly solvable for some* $s \in \{\pm 1\}^n$.

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*The interval system*

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$$(D_s A(p))x \leq D_s b(p), \quad p \in \boldsymbol{p}$$

*is strongly solvable for each $s \in \{\pm 1\}^m$.*

**Theorem**

*The interval system*

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# AE Solutions of Parametric Systems

Let $A(p), b(p)$ be functions of $p \in \boldsymbol{p} \in \mathbb{IR}^K$.

Let each interval $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K$ be associated either with $\forall$, or $\exists$.

**Theorem**

*The system*

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## Theorem

*The system*

$$A(p)x \le b(p), \quad p \in \boldsymbol{p}$$

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*has an AE solution for some $s \in \{\pm 1\}^n$.*

# Conclusion

Any other examples?

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