On the optimal solution set in interval linear programming

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Interval-Valued Problems

Sources of Intervals

- physical constants
  - $[9.78, 9.82]\, ms^{-2}$ for the gravitational acceleration
- mathematical constants
  - $\pi \in [3.14159265358979323\,84, 3.14159265358979323\,85]$.
- measurement errors
  - temperature measured $23°\mathrm{C} \pm 1°\mathrm{C}$
- discretization
  - time is split in days
  - temperature during the day in $[18, 29]°\mathrm{C}$ for Brno in June
- missing data
  - What was the temperature in Brno on June 9, 1996?
  - Very probably in $[10, 40]°\mathrm{C}$.

Interval Linear Programming

Interval Linear Programming Problem

A family of linear programs

$$\min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

where $c \in \boldsymbol{c} = [\underline{c}, \overline{c}]$, $b \in \boldsymbol{b} = [\underline{b}, \overline{b}]$, and $A \in \boldsymbol{A} = [\underline{A}, \overline{A}]$.

Related Problems

- feasibility, boundedness, etc. for some / all instances,
- set of optimal values,
- set of optimal solutions.

Optimal Solution Set

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} \mathcal{S}(A, b, c).$$

Typical Problem

Find a tight enclosure to $\mathcal{S}$.

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, and $c \in \boldsymbol{c}$ such that

$$Ax = b, \ x \geq 0, \ A^T y \leq c, \ c^T x = b^T y,$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

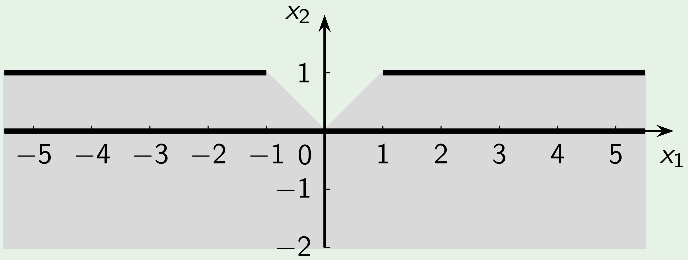
Optimal Solution Set – Properties

Example (Garajová, 2016)

The optimal solution set may be disconnected and nonconvex.

Consider the interval LP problem

$$\max\ x_2 \ \text{ subject to } \ [-1,1]x_1 + x_2 \leq 0,\ x_2 \leq 1.$$



Optimal Solution Set – Properties

Theorem (Garajová, H., 2016)

Assume the set of optimal solutions of the dual interval problem

$$\max\ \boldsymbol{b}^T y \ \ \textit{subject to}\ \ \boldsymbol{A}^T y \le \boldsymbol{c},\ \ y \in \mathbb{R}^m$$

is bounded. Then the set of optimal solutions $\mathcal{S}$ is closed.

Theorem (Garajová, H., 2016)

The set of optimal solutions of the interval linear program (with real A)

$$\min\ \boldsymbol{c}^T x \ \ \textit{subject to}\ \ Ax = \boldsymbol{b},\ \ x \ge 0$$

is a union of at most 2^n convex polyhedra.

Basis Stability

Definition

The interval linear programming problem

$$\min \boldsymbol{c}^T x \quad \text{subject to} \quad \boldsymbol{A}x = \boldsymbol{b},\ x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\underline{f} = \min\ \underline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,$$

$$\overline{f} = \max\ \overline{c}_B^T x \quad \textit{subject to} \quad \underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0.$$

Under the unique B-stability, the set of all optimal solutions reads

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

(Otherwise each realization has at least one optimal solution in this set.)

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C1

- C1 says that $\boldsymbol{A}_B$ is regular;
- co-NP-hard problem;
- Beeck's sufficient condition: $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$.

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C2

- C2 says that the solution set to $\boldsymbol{A}_B x_B = \boldsymbol{b}$ lies in $\mathbb{R}^n_+$;
- sufficient condition: check of some enclosure to $\boldsymbol{A}_B x_B = \boldsymbol{b}$.

Basis Stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$.

Condition C3

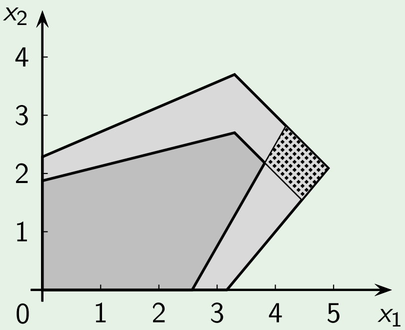
- C3 says that $\boldsymbol{A}_N^T y \leq \boldsymbol{c}_N,\ \boldsymbol{A}_B^T y = \boldsymbol{c}_B$ is strongly feasible;
- co-NP-hard problem;
- <u>sufficient</u> condition:
  $(\boldsymbol{A}_N^T)\boldsymbol{y} \leq \underline{c}_N$, where $\boldsymbol{y}$ is an enclosure to $\boldsymbol{A}_B^T y = \boldsymbol{c}_B$.

Example

Example

Consider an interval linear program

$$\max\ ([5,6],[1,2])^T x \ \text{ s.t. } \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix},\ x \geq 0.$$



- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area

Basis Stability – Interval Right-Hand Side

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$ for each $b \in \boldsymbol{b}$.

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Condition C1

- C1 and C3 are trivial
- C2 is simplified to

$$\underline{A_B^{-1} \boldsymbol{b}} \geq 0,$$

  which is easily verified by interval arithmetic
- overall complexity: polynomial

Basis Stability – Interval Objective Function

Interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$ for each $c \in \boldsymbol{c}$

Condition C1

- C1 and C2 are trivial
- C3 is simplified to

$$A_N^T y \leq \boldsymbol{c}_N,\ A_B^T y = \boldsymbol{c}_B$$

or,

$$\overline{(A_N^T A_B^{-T})\boldsymbol{c}_B} \leq \underline{c}_N.$$

- overall complexity: polynomial

Applications

Real-Life Applications

- Transportation problems with uncertain demands, suppliers, and/or costs.
- Networks flows with uncertain capacities.
- Diet problems with uncertain amounts of nutrients in foods.
- Portfolio selection with uncertain rewards.
- Matrix games with uncertain payoffs.

Technical Applications

- Tool for global optimization.
- Measure of sensitivity of linear programs.

Applications – Diet Problem

Example (Stigler's Nutrition Model)

`http://www.gams.com/modlib/libhtml/diet.htm`.

- $n = 20$ different types of food,
- $m = 9$ nutritional demands,
- a_{ij} is the the amount of nutrient j contained in one unit of food i,
- b_i is the required minimal amount of nutrient j,
- c_j is the price per unit of food j,
- minimize the overall cost

The model reads

$$\min c^T x \quad \text{subject to} \quad Ax \geq b, \; x \geq 0.$$

The entries a_{ij} are not stable!

Applications – Diet Problem

Example (Stigler's Nutrition Model (cont.))

Nutritive value of foods (per dollar spent)

	calorie (1000)	protein (g)	calcium (g)	iron (mg)	vitamin-a (1000iu)	vitamin-b1 (mg)	vitamin-b2 (mg)	niacin (mg)	vitamin-c (mg)
wheat	44.7	1411	2.0	365		55.4	33.3	441	
cornmeal	36	897	1.7	99	30.9	17.4	7.9	106	
cannedmilk	8.4	422	15.1	9	26	3	23.5	11	60
margarine	20.6	17	.6	6	55.8	.2			
cheese	7.4	448	16.4	19	28.1	.8	10.3	4	
peanut-b	15.7	661	1	48		9.6	8.1	471	
lard	41.7				.2		.5	5	
liver	2.2	333	.2	139	169.2	6.4	50.8	316	525
porkroast	4.4	249	.3	37		18.2	3.6	79	
salmon	5.8	705	6.8	45	3.5	1	4.9	209	
greenbeans	2.4	138	3.7	80	69	4.3	5.8	37	862
cabbage	2.6	125	4	36	7.2	9	4.5	26	5369
onions	5.8	166	3.8	59	16.6	4.7	5.9	21	1184
potatoes	14.3	336	1.8	118	6.7	29.4	7.1	198	2522
spinach	1.1	106		138	918.4	5.7	13.8	33	2755
sweet-pot	9.6	138	2.7	54	290.7	8.4	5.4	83	1912
peaches	8.5	87	1.7	173	86.8	1.2	4.3	55	57
prunes	12.8	99	2.5	154	85.7	3.9	4.3	65	257
limabeans	17.4	1055	3.7	459	5.1	26.9	38.2	93	
navybeans	26.9	1691	11.4	792		38.4	24.6	217	

Applications – Diet Problem

Example (Stigler's Nutrition Model (cont.))

If the entries a_{ij} are known with 10% accuracy, then

- the problem is not basis stable
- the minimal cost ranges in $[0.09878, 0.12074]$,
- the interval enclosure of the solution set is

$$[0, 0.0734],\ [0, 0.0438],\ [0, 0.0576],\ [0, 0.0283],\ [0, 0.0535],\ [0, 0.0315],\ [0, 0.0339],$$
$$[0, 0.0300],\ [0, 0.0246],\ [0, 0.0337],\ [0, 0.0358],\ [0, 0.0387],\ [0, 0.0396],\ [0, 0.0429],$$
$$[0, 0.0370],\ [0, 0.0443],\ [0, 0.0290],\ [0, 0.0330],\ [0, 0.0472],\ [0, 0.1057].$$

If the entries a_{ij} are known with 1% accuracy, then

- the problem is basis stable
- the minimal cost ranges in $[0.10758, 0.10976]$,
- the interval hull of the solution set is

$$x_1 = [0.0282, 0.0309],\ x_8 = [0.0007, 0.0031],\ x_{12} = [0.0110, 0.0114],$$
$$x_{15} = [0.0047, 0.0053],\ x_{20} = [0.0600, 0.0621].$$

Conclusion

Open Problems

- More about topology of the optimal solution set $\mathcal{S}$ (connectivity, closedness, convexity, etc.),
- characterization of $\mathcal{S}$,
- tight approximation of $\mathcal{S}$.

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