Interval Convex Quadratic Programming Problems in a General Form

Milan Hladík[1,2] Michal Černý[1] Jan Pelikán[1]

[1] Department of Econometrics,
University of Economics, Prague, Czech Republic

[2] Department of Applied Mathematics
Charles University in Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

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Problem Formulation

Convex quadratic programming (CQP)

Consider a CQP problem in a general form

$$\min \begin{pmatrix} x^T & y^T \end{pmatrix} \begin{pmatrix} P & R \\ R^T & S \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a^T x + c^T y$$

$$\text{subject to} \quad Ax + By = b, \; Cx + Dy \leq d, \; x \geq 0,$$

where $\left(\begin{smallmatrix} P & R \\ R^T & S \end{smallmatrix}\right)$ is positive semidefinite.

Interval CQP

Let input coefficients vary in certain intervals $A \in [\underline{A}, \overline{A}] \equiv \boldsymbol{A}, \ldots$

- Compute $\underline{f}$, the smallest optimal value subject to $A \in \boldsymbol{A}, \ldots$
- Compute $\overline{f}$, the largest optimal value subject to $A \in \boldsymbol{A}, \ldots$

Theorem

Computation of $\underline{f}$ and $\overline{f}$ are NP-hard problems.

Lower bound $\underline{f}$

Theorem

We can reduce computation of $\underline{f}$ to 2^n standard CQP's, where n is the number of free variables.

Corollary

1. *For the interval CQP*

$$\min\ x^T \boldsymbol{P} x + \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A} x \leq \boldsymbol{b},\ x \geq 0,$$

   we have

$$\underline{f} = \min\ x^T \underline{P} x + \underline{c}^T x \ \text{ subject to } \ \underline{A} x \leq \overline{b},\ x \geq 0.$$

2. *For the interval CQP*

$$\min\ x^T \boldsymbol{P} x + \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A} x = \boldsymbol{b},\ x \geq 0,$$

   we have

$$\underline{f} = \min\ x^T \underline{P} x + \underline{c}^T x \ \text{ subject to } \ \underline{A} x \leq \overline{b},\ \overline{A} x \geq \underline{b},\ x \geq 0.$$

Upper bound $\overline{f}$

Theorem

We can reduce computation of $\overline{f}$ to 2^{m+n} standard CQP's, where m is the number of equations and n is the number of free variables.

Corollary

For the interval CQP

$$\min \ x^T \boldsymbol{P} x + \boldsymbol{c}^T x \ \ \textit{subject to} \ \ \boldsymbol{A} x \le \boldsymbol{b}, \ x \ge 0,$$

we have

$$\overline{f} = \min \ x^T \overline{P} x + \overline{c}^T x \ \ \textit{subject to} \ \ \overline{A} x \le \underline{b}, \ x \ge 0.$$

Open problem

Exponentiality w.r.t. n necessary?

Application: distance of polytopes

Distance of polytopes

Let

- columns of C constitute vertices of the first polytope,
- columns of $-D$ constitute vertices of the second polytope.

The minimal distance (its square) between them can be solved by CQP

$$f(C, D) = \min\ z^T z\ \text{ subject to }\ z = Cx + Dy,\ e^T x = e^T y = 1,\ x, y \geq 0.$$

Remark

The polytopes are intersecting if and only if $f(C, D) = 0$.

Uncertain data

Suppose that $C \in \boldsymbol{C}$ and $D \in \boldsymbol{D}$. Compute:

$$\underline{f} := \min\ f(C, D)\ \text{ subject to }\ C \in \boldsymbol{C},\ D \in \boldsymbol{D},$$

$$\overline{f} := \max\ f(C, D)\ \text{ subject to }\ C \in \boldsymbol{C},\ D \in \boldsymbol{D}.$$

Application: distance of polytopes

Minimal separation distance

Reduction to one CQP problem:

$$\underline{f} := \min \ z^T z \quad \text{subject to} \ \ \underline{C}x + \underline{D}y \le z \le \overline{C}x + \overline{D}y, \ e^T x = e^T y = 1, \ x, y \ge 0.$$

Maximal separation distance (exp. w.r.t. dimension, not vertices)

Reduction to 2^m CQP problems:

$$\overline{f} = \max_{s \in \{\pm 1\}^m} f_s,$$

where

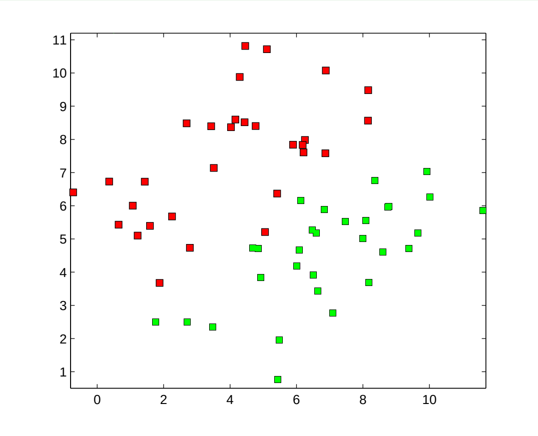
$$f_s := \min \ z^T z \ \ \text{subject to} \ \ z = C^s x + D^s y, \ e^T x = e^T y = 1, \ x, y \ge 0,$$

and

$$C^s_{ij} = \begin{cases} \underline{C}_{ij} & \text{if } s_i = 1, \\ \overline{C}_{ij} & \text{if } s_i = -1, \end{cases} \qquad D^s_{ij} = \begin{cases} \underline{D}_{ij} & \text{if } s_i = 1, \\ \overline{D}_{ij} & \text{if } s_i = -1. \end{cases}$$

Application: distance of polytopes

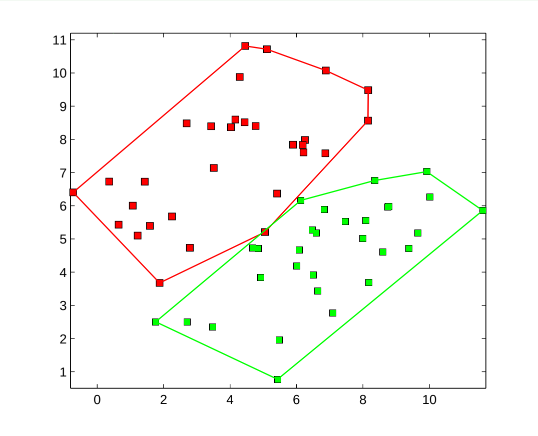
Example



- Random data in $\mathbb{R}^2$; both polytopes consisting of 30 vertices.

Application: distance of polytopes

Example



- The convex hull for the midpoint values.
- We computed $\underline{f} = 0$ and $\overline{f} = 0.6509$.

Conclusion

Interval convex quadratic programming

- General form (equations and inequalities, free and nonnegative variables)
- Explicit formulae for the optimal value bounds (easy cases identified).
- Approximation of $\underline{f}$ and $\overline{f}$.
- Open problem: Does computation of $\overline{f}$ depend exponentially on the number of free variables?

Future work

- Extension to quadratically constrained problems.

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