On relation between P-matrices and regularity of interval matrices

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P-matrix

Definition

$A \in \mathbb{R}^{n \times n}$ is a P-matrix if all its principal minors are positive.

Key application

A linear complementarity problem

$$q + Mx \geq 0, \quad x \geq 0, \quad (q + Mx)^T x = 0.$$

has a unique solution for each q if and only if M is a P-matrix.

Theorem (Fiedler & Pták, 1962)

A matrix $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if for each vector $x \neq 0$ there is i such that $x_i (Ax)_i > 0$.

P-matrix

Definition

$A \in \mathbb{R}^{n \times n}$ is a P-matrix if all its principal minors are positive.

Complexity

Checking P-matrix property is co-NP-hard (Coxson, 1994).

Sub-classes

Special (easily identifiable) sub-classes of P-matrices (Peña, 2001, Tsatsomeros, 2004):

- positive definite matrices
- M-matrices ($a_{ij} \leq 0 \ \forall i, j$ and $A^{-1} \geq 0$),
- B-matrices ($\sum_{k=1}^{n} a_{ik} > 0$ and $\frac{1}{n} \sum_{k=1}^{n} a_{ik} > a_{ij}$ for $j \neq i$),
- H-matrices with positive diagonal entries
  (A is an H-matrix if $\langle A \rangle$ is an M-matrix, where $\langle A \rangle_{ii} = |a_{ii}|$ and $\langle A \rangle_{ij} = -|a_{ij}|$, $i \neq j$).

Interval matrix

Notation

- An interval matrix $\mathbf{A}$ is defined as

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\},$$

- The center and radius of $\mathbf{A}$ are respectively defined as

$$\mathbf{A}_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad \mathbf{A}_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

- The set of all m-by-n interval matrices is denoted by $\mathbb{IR}^{m\times n}$.

Definition

$A \in \mathbb{IR}^{n\times n}$ is regular if every $A \in \mathbf{A}$ is nonsingular.

Properties

- Checking regularity is co-NP-hard (Poljak & Rohn, 1988).
- Forty equivalent characterizations (Rohn, 2009).

Known relations

Theorem (Rohn, 1989)

Let $\mathbf{A} \in \mathbb{IR}^{n \times n}$ be regular. Then $A_1^{-1}A_2$ is a P-matrix for each $A_1, A_2 \in \mathbf{A}$.

Theorem (Rohn, 1989)

An interval matrix $\mathbf{A} \in \mathbb{IR}^{n \times n}$ is regular if and only if for each $y \in \{\pm 1\}^n$ the matrix $A_c - D_y A_\Delta$ is nonsingular and $(A_c - D_y A_\Delta)^{-1}(A_c + D_y A_\Delta)$ is a P-matrix.

Theorem (Rump, 2003)

Let $A \in \mathbb{IR}^{n \times n}$ with $A - I$ and $A + I$ nonsingular. Then A is a P-matrix if and only if $[(A - I)^{-1}(A + I) - I, \, (A - I)^{-1}(A + I) + I]$ is regular.

New relations

Theorem

Let $A \in \mathbb{R}^{n\times n}$. If $\alpha > 0$ is sufficiently small, then $P := \alpha A$ is a P-matrix if and only if $[(I-P)^{-1} - I, (I-P)^{-1} + I]$ is regular.

Open problem

Tight estimation of α.

Theorem

Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ with A_c nonsingular. Then $\mathbf{A}$ is regular if and only if $I - A_c^{-1} D_y A_\Delta D_z$ is a P-matrix for each $y, z \in \{\pm 1\}^n$.

Sufficient conditions for P-matrices

Theorem

$A \in \mathbb{R}^{n\times n}$ is a P-matrix if $A - I$ and $A + I$ are nonsingular and $\rho(|(A+I)^{-1}(A-I)|) < 1$.

Proof.

Rump's theorem and sufficient condition for regularity of $[(A-I)^{-1}(A+I) - I, (A-I)^{-1}(A+I) + I]$. □

Theorem

The matrix $A \in \mathbb{R}^{n\times n}$ is a P-matrix provided for $I - \alpha A$ is nonsingular and $\rho(|I - \alpha A|) < 1$ for some $\alpha > 0$.

Remark

Weaker than checking H-matrix property.

Interval P-matrices

Definition

An interval matrix $A \in \mathbb{IR}^{n\times n}$ is called *an interval P-matrix* if each $A \in \mathbf{A}$ is a P-matrix.

Theorem (Białas and Garloff, 1984)

$\mathbf{A} \in \mathbb{IR}^{n\times n}$ is an interval P-matrix if and only if $A_c - D_z A_\Delta D_z$ is a P-matrix for each $z \in \{\pm 1\}^n$.

Corollary

Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ such that $A_c = I_n$. Then $\mathbf{A}$ is an interval P-matrix if and only if $\underline{A}$ is a P-matrix.

Open problem

How to employ for checking interval P-property?

References

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