What computational complexity theory tells us about (some) statistical problems

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Part I. Computational complexity: A pair of examples

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Introduction: Computational complexity

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- Many further and finer questions (weak/strong polynomiality, pseudopolynomiality, randomized computing, reductions among problems, polynomial-time hierarchy, space (memory) complexity . . .)

Examples

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- **And statistics ??**

An example of our previous work: c-optimal design

Formulation. Consider the linear regression model $y = X\beta + \varepsilon$, where ε_i are $N(0, \sigma^2)$ independent. We are given a finite experimental domain Ξ and $c \neq 0$. Two natural problems:

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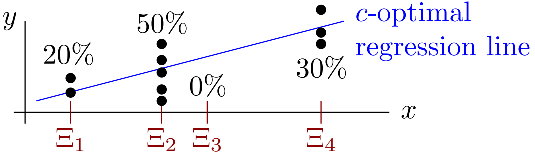

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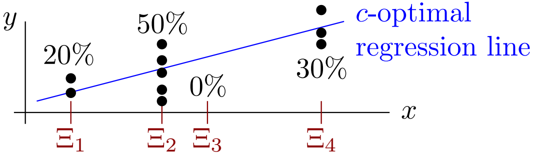

- **ExactDesign**: Given n, find the n-exact c-optimal design over Ξ.
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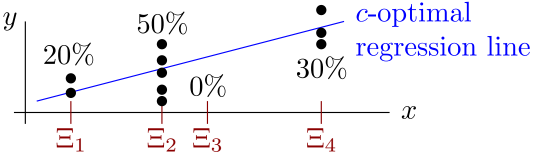

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- Are these problems computational easy or hard?

c-optimal design (contd.)

Some results (Černý & Hladík, *Comput Optim Appl*, 2012):

- **ExactDesign** is NP-hard.
- **ApproxDesign** is computable in polynomial time, but it is P-complete — the "hardest" among all efficiently computable problems.
- **ApproxDesign** cannot be efficiently parallelized.
- **ApproxDesign** is equivalent to general linear programming.
  - The message: *Do not try to design "good" algorithms for the problem. If you try to, then you will be competing against interior point algorithms and it is not easy to defeat them. But if you succeed, you'll be truly famous in optimization.*

Part II. Computational complexity and analysis of one-dimensional interval data

One-dimensional interval data: a model

Assumptions.

- Let $x_1, \ldots, x_n$ be a dataset; for example, let the data be a random sample from a distribution Φ. The dataset is unobservable.

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- We are given a statistic $S(x_1, \ldots, x_n)$ and we want to determine/estimate its value, distribution, or other properties, using only the observable intervals $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

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- **Now:** the appropriate toolbox depends on whether we can make further assumptions on the distribution of $(x, \mathbf{x})$.
  - *For example, in which practical situations can we assume that x is uniformly distributed on $\mathbf{x}$ and when we cannot?*

The possibilistic approach

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- In this lecture: our only knowledge about $(x, \mathbf{x})$ is $x \in \mathbf{x}$ a.s. **Nothing more.**
- Then, given a statistic S, the only information we can infer about S from the observable intervals $\mathbf{x}$ is the pair of tight bounds

$$\overline{S} = \max\{S(\xi) : \xi \in \mathbf{x}\},$$
$$\underline{S} = \min\{S(\xi) : \xi \in \mathbf{x}\},$$

clearly satisfying

$$\underline{S} \leqslant S(x) \leqslant \overline{S} \ \text{ a.s.}$$

- **Remark.** In econometrics, partial knowledge about the distribution $(x, \mathbf{x})$ is referred to as partial identification: see the survey paper E. Tamer, **Partial identification in econometrics**, Annual Review of Economics 2 (2010), pp. 167–195.
- Also many papers in Econometrica and other journals.

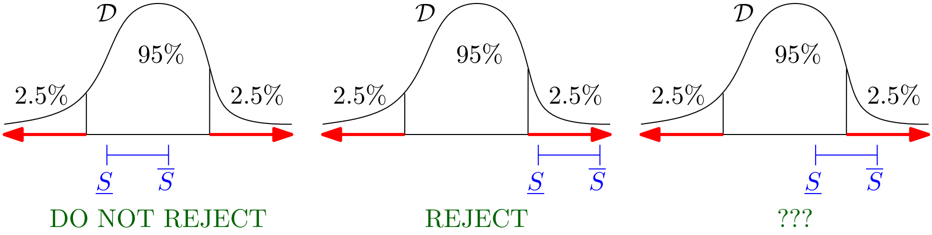
The core problem is:

- Given a statistic $S(x_1, \ldots, x_n)$ and the intervals $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, is it computationally easy or difficult to determine

$$\overline{S} = \max\{S(\xi) : \xi \in \mathbf{x}\} \quad \text{and} \quad \underline{S} = \min\{S(\xi) : \xi \in \mathbf{x}\} \ ?$$

Example: Test statistics

- We are to test a null hypothesis (H_0) against an alternative A using a test statistic S
- Let $\mathcal{D}$ be the distribution of S under H_0
- Given the intervals $\mathbf{x}_1, \ldots, \mathbf{x}_n$: if we can compute $\underline{S}, \overline{S}$, then we can make at least partial conclusions:



Sample variance & complexity

$$\overline{s^2} = \max \left\{ \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right)^2 \ : \ x \in \mathbf{x} \right\},$$

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- **Observation:** $\underline{s^2} \to$ CQP $\to$ weakly polynomial time
- **Ferson et al.:** a strongly polynomial algorithm $O(n^2)$
- **Unfortunately:** $\overline{s^2}$ is NP-hard
- **Even worse:** $\overline{s^2}$ is NP-hard to approximate with an **arbitrary** absolute error
  - *The message: if somebody claims that (s)he can design an efficient algorithm for computing $\overline{s^2}$ with an error at most ± 1000, then (s)he has proved $P = NP$ and will get the \$1M award from the Clay Math Institute...*

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- **Following Option 2, we'll try to show that the situation isn't so catastrophic.**

An algorithm by Ferson et al.

- **Notation:** for an interval $\mathbf{x} = [\underline{x}, \overline{x}]$, define

$$x^C := \tfrac{1}{2}(\overline{x} + \underline{x}) \text{ (center)}, \quad x^\Delta := \tfrac{1}{2}(\overline{x} - \underline{x}) \text{ (radius)}$$

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- Another formulation: If there is no k-tuple of indices $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that

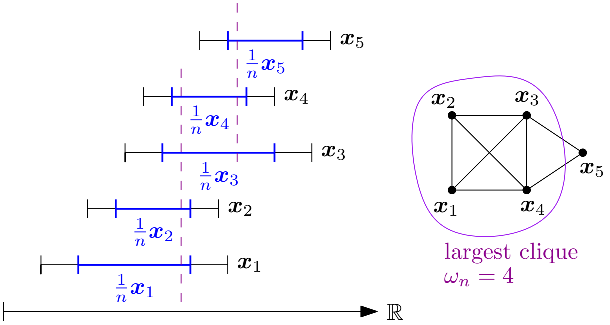
$$\bigcap_{\ell \in \{i_1, \dots, i_k\}} \tfrac{1}{n}\mathbf{x}_\ell \neq \emptyset,$$

  then $\overline{s^2}$ can be computed in time $O(p(n) \cdot 2^k)$, where p is a polynomial.

Computation of $\overline{s^2}$ & Ferson et al. (contd.)

Graph-theoretic reformulation: Let $G_n(V_n, E_n)$ be the interval graph over $\frac{1}{n}\mathbf{x}_1, \ldots, \frac{1}{n}\mathbf{x}_n$:

- Vertices: V_n = set of the narrowed intervals $\frac{1}{n}\mathbf{x}_1, \ldots, \frac{1}{n}\mathbf{x}_n$
- Edges: $\{i, j\} \in E$ $(i \neq j)$ iff $\frac{1}{n}\mathbf{x}_i \cap \frac{1}{n}\mathbf{x}_j \neq \emptyset$
- Let ω_n be the size of the largest clique of G_n. Now: the algorithm works in time $O(p(n) \cdot 2^{\omega_n})$.



Remark. The worst case is bad — e.g. when $\mathbf{x}_1^C = \mathbf{x}_2^C = \cdots = \mathbf{x}_n^C$. (Such instances result from the NP-hardness proof.)

But: What if the data are generated by a random process? Then, do the "ugly" instances occur frequently, or only rarely?

Simulations

Assumption: The centers and radii of intervals $\mathbf{x}_i$ are generated by a "reasonable" random process:

- Centers $\mathbf{x}_i^C$: sampled from a "reasonable" distribution (continuous, finite variance) — uniform, normal, exp, ...
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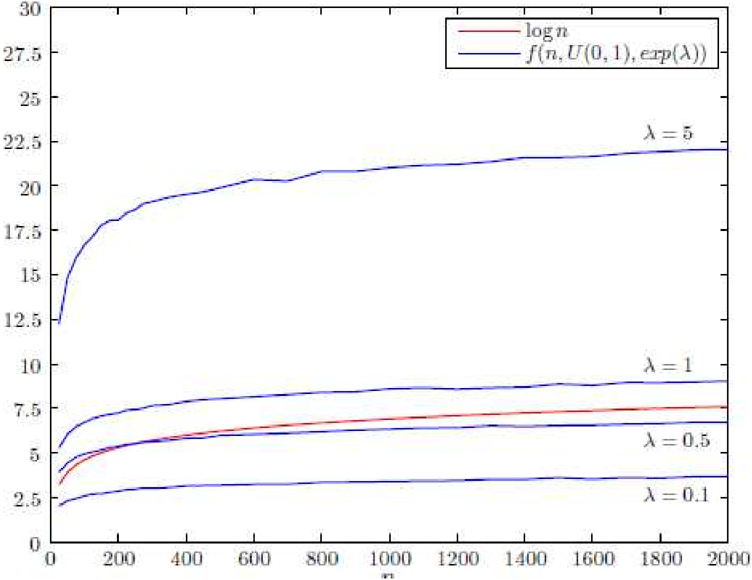
- Simulations show Sokol's conjecture: The clique is **logarithmic** on average!
- If indeed $\mathrm{E}\omega_n = O(\log n)$, then the average computation time is

$$O(p(n) \cdot 2^{\omega_n}) = O(p(n) \cdot 2^{O(\log n)}) = \text{polynomial}(n).$$

- Thus: The algorithm is polynomial on average (even if exponential in the worst case).

Sokol's conjecture



Furthermore: It seems that $\mathrm{var}(\omega_n) = O(1)$ ("Sokol's conjecture II").

- Say, for simplicity, that indeed $\mathrm{E}\omega_n = \log n$. By Chebyshev's inequality we get:

$$\Pr[\omega_n \geqslant \log n + \underbrace{10\sqrt{\mathrm{var}(\omega_n)}}_{=:K \text{ (constant)}}] \leqslant 1\%.$$

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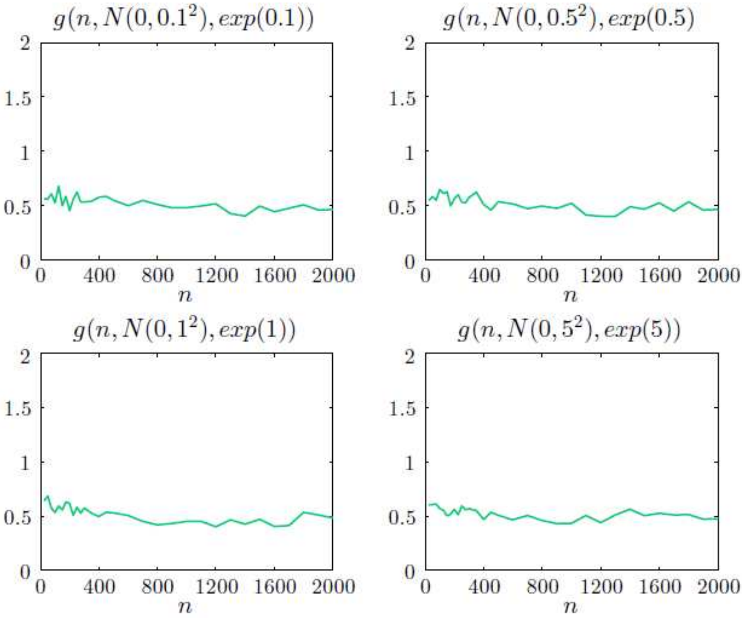
- Thus: in 99% cases, the algorithm of Ferson et al. works in time at most

$$p(n) \cdot 2^{K + \log n},$$

where K does not grow with n.

Sokol's conjecture II



A pair of remarks

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- We have random intersection (interval) graphs and we need to estimate the average size of the largest clique and its variance
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Further good news:

- $\overline{s^2}$ is computable pseudopolynomially
- Main message: although NP-hard in theory, $\overline{s^2}$ is efficiently computable "almost always" (in the probabilistic setup) — hard instances are rare
- A nice interdisciplinary problem: statistical motivation, interval-theoretic and graph-theoretic methods
- Some ideas can be generalized for other statistics which are known to be NP-hard, e.g. the F-ratio

$$F = \frac{\text{sample variance of } x_1, \ldots, x_{n/2}}{\text{sample variance of } x_{(n/2)+1}, \ldots, x_n}.$$

Conclusions & further reading

- The coefficient of variation (t-ratio) has been studied in our paper Černý M. and Hladík M. Complexity of computation and approximation of the t-ratio over one-dimensional interval data. *Comput Stat Data Anal* 80, 2014, 26–43.
- Many further results can be found in the book H. Nguyen et al. *Computing statistics under interval and fuzzy uncertainty. Applications to computer science and engineering.* Vol. 393 of Studies in Computational Intelligence, Springer, Berlin, 2012.
- We have further results of this kind in linear regression, see e.g. our preprint Hladík M. and Černý M. Linear regression with interval data: Computational issues (available from `http://nb.vse.cz/~cernym/ilr.pdf`).

Thank you!