Positive Semidefiniteness and Positive Definiteness of a Linear Parametric Interval Matrix

Milan Hladík

Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic
http://kam.mff.cuni.cz/~hladik/

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Motivation

Why positive (semi)definiteness of interval matrices?

In global optimization, for convexity checking:

- If a function is convex on a box, then a stationary point is a minimum.
- If a function is convex nowhere on a box, and the box is inside the feasible set, then there is no minimum inside.

Also:

- Hurwitz stability of dynamical systems.
- Schur stability of dynamical systems.

Notation

Interval Matrix

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A^c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $m \times n$ interval matrices: $\mathbb{IR}^{m \times n}$.

A Symmetric Interval Matrix

$$\mathbf{A}^S := \{A \in \mathbf{A} : A = A^T\}.$$

Without loss of generality assume that $\underline{A} = \underline{A}^T$, $\overline{A} = \overline{A}^T$, and $\mathbf{A}^S \neq \emptyset$.

Positive Semidefiniteness and Positive Definiteness

$\mathbf{A}^S$ is *positive (semi)definite* if every $A \in \mathbf{A}^S$ is positive (semi)definite.

Theorem (Rohn, 1994)

The following are equivalent

1. $\mathbf{A}^S$ *is positive semidefinite,*
2. $A^c - \operatorname{diag}(z) A^\Delta \operatorname{diag}(z)$ *is positive semidefinite* $\forall z \in \{\pm 1\}^n$,
3. $x^T A^c x - |x|^T A^\Delta |x| \geq 0$ *for each* $x \in \mathbb{R}^n$.

Theorem (Rohn, 1994)

The following are equivalent

1. $\mathbf{A}^S$ *is positive definite,*
2. $A^c - \operatorname{diag}(z) A^\Delta \operatorname{diag}(z)$ *is positive definite for each* $z \in \{\pm 1\}^n$,
3. $x^T A^c x - |x|^T A^\Delta |x| > 0$ *for each* $0 \neq x \in \mathbb{R}^n$,
4. A^c *is positive definite and* $\mathbf{A}$ *is regular.*

Complexity

Theorem (Nemirovskii, 1993)

Checking positive semidefiniteness of $\mathbf{A}^S$ is co-NP-hard.

Theorem (Rohn, 1994)

Checking positive definiteness of $\mathbf{A}^S$ is co-NP-hard.

Theorem (Jaulin and Henrion, 2005)

Checking whether there is a positive semidefinite matrix in $\mathbf{A}^S$ is a polynomial time problem.

Proof.

By reduction to semidefinite programming. □

Sufficient Conditions

Theorem

1. $\mathbf{A}^S$ *is positive semidefinite if* $\lambda_{\min}(A^c) \geq \rho(A^\Delta)$.
2. $\mathbf{A}^S$ *is positive definite if* $\lambda_{\min}(A^c) > \rho(A^\Delta)$.
3. $\mathbf{A}^S$ *is positive definite if* A^c *is positive definite and* $\rho(|(A^c)^{-1}|A^\Delta) < 1$.

Proof.

1. $\mathbf{A}^S$ is positive semidefinite iff $\lambda_{\min}(A) \geq 0 \ \forall A \in \mathbf{A}^S$.
   Now, employ the smallest eigenvalue set enclosure
   $$\lambda_{\min}(A) \in [\lambda_{\min}(A^c) - \rho(A^\Delta), \lambda_{\min}(A^c) + \rho(A^\Delta)] \quad \forall A \in \mathbf{A}^S.$$
2. Analogous.
3. Use Beeck's sufficient condition for regularity of $\mathbf{A}$. $\square$

Application: Convexity Testing

Theorem

A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ *is convex on* $\mathbf{x} \in \mathrm{I\!R}^n$ *iff its Hessian* $\nabla^2 f(x)$ *is positive semidefinite* $\forall x \in int\, \mathbf{x}$.

Corollary

A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ *is convex on* $\mathbf{x} \in \mathrm{I\!R}^n$ *if* $\nabla^2 f(\mathbf{x})$ *is positive semidefinite.*

Application: Convexity Testing

Example

Let

$$f(x,y,z) = x^3 + 2x^2y - xyz + 3yz^2 + 8y^2,$$

on $x \in \mathbf{x} = [2,3]$, $y \in \mathbf{y} = [1,2]$ and $z \in \mathbf{z} = [0,1]$. The Hessian of f reads

$$\nabla^2 f(x,y,z) = \begin{pmatrix} 6x+4y & 4x-z & -y \\ 4x-z & 16 & -x+6z \\ -y & -x+6z & 6y \end{pmatrix}$$

Evaluation the Hessian matrix by interval arithmetic results in

$$\nabla^2 f(\mathbf{x},\mathbf{y},\mathbf{z}) \subseteq \begin{pmatrix} [16,26] & [7,12] & -[1,2] \\ [7,12] & 16 & [-3,4] \\ -[1,2] & [-3,4] & [6,12] \end{pmatrix}$$

Now, both sufficient conditions for positive definiteness succeed.

Thus, we can conclude that f si convex on the interval domain.

Parametric Interval Matrices

Parametric Interval Matrix

Consider

$$A(p) = \sum_{k=1}^{K} A^{(k)} p_k,$$

where $A^{(1)}, \ldots, A^{(K)} \in \mathbb{R}^{n \times n}$ are fixed symmetric matrices and $p_1, \ldots, p_K$ are parameters varying respectively in $\mathbf{p}_1, \ldots, \mathbf{p}_K \in \mathbb{IR}$.

Definition

- $A(p)$, $p \in \mathbf{p}$, is *strongly positive (semi)definite* if $A(p)$ is positive (semi)definite for each $p \in \mathbf{p}$.
- It is *weakly positive (semi)definite* if $A(p)$ is positive (semi)definite for at least one $p \in \mathbf{p}$.

Parametric Interval Matrices

Relaxation

Evaluation $A(\mathbf{p}) = \sum_{k=1}^{K} A^{(k)} \mathbf{p}_k$ by interval arithmetic

- encloses the set of matrices $A(p)$, $p \in \mathbf{p}$,
- may lead to loss of strong positive (semi-)definiteness.

Example

Let

$$A(p) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} p, \quad p \in \mathbf{p} = [0, 1].$$

This parametric matrix is strongly positive semidefinite, but its relaxation

$$A(\mathbf{p}) = \begin{pmatrix} [0, 1] & [0, 1] \\ [0, 1] & [0, 1] \end{pmatrix}$$

is not as it contains, e.g., the indefinite matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Strong Positive Semidefiniteness

Theorem

The following are equivalent:

(1) $A(p)$ *is positive semidefinite for each* $p \in \mathbf{p}$,

(2) $A(p)$ *is positive semidefinite for each* p *such that* $p_k \in \{\underline{p}_k, \overline{p}_k\}$ $\forall k$,

(3) $x^T A(p^c)x - \sum_{k=1}^{K} |x^T A^{(k)} x| p_k^{\Delta} \geq 0$ *for each* $x \in \mathbb{R}^n$.

- It reduces the problem to checking positive semidefiniteness of 2^K real matrices.
- The number can be further decreased in some cases.

Theorem

(1) *If* $A^{(i)}$ *is positive semidefinite for some* i, *then we can fix* $p_i := \underline{p}_i$ *for checking strong positive semidefiniteness.*

(2) *If* $A^{(i)}$ *is negative semidefinite for some* i, *then we can fix* $p_i := \overline{p}_i$ *for checking strong positive semidefiniteness.*

Strong Positive Semidefiniteness – Sufficient Condition

Theorem

For each k, split $A^{(k)} = A_1^{(k)} - A_2^{(k)}$ such that both $A_1^{(k)}, A_2^{(k)}$ are positive semidefinite. Then $A(p)$, $p \in \mathbf{p}$, is strongly positive semidefinite if

$$\sum_{k=1}^{K} \left(A_1^{(k)} \underline{p}_k - A_2^{(k)} \overline{p}_k \right)$$

is positive semidefinite.

How to Do the Splitting

1. Let $A^{(k)} = Q \Lambda Q^T$ be a spectral decomposition of $A^{(k)}$.
2. Let Λ^+ be the positive part of Λ.
3. Let Λ^- be the negative part of Λ.
4. Then $A^{(k)} = Q\Lambda Q^T = Q\Lambda^+ Q^T - Q\Lambda^- Q^T$ and both $Q\Lambda^+ Q^T$, $Q\Lambda^- Q^T$ are positive semidefinite.

Overall cost: $K + 1$ spectral decompositions.

Weak Positive Semidefiniteness

Theorem

Checking weak positive semidefiniteness is a polynomial problem.

Proof.

By reduction to semidefinite programming. Let $M(p)$ be the block diagonal matrix with blocks

$$A(p),\ p_1 - \underline{p}_1,\ \ldots,\ p_K - \underline{p}_K,\ \overline{p}_1 - p_1,\ \ldots,\ \overline{p}_K - p_K.$$

- All entries of $M(p)$ depends affinely on variables p.
- Positive definiteness of $M(p)$ is equivalent to positive definiteness of $A(p)$ and feasibility of variables $p \in \mathbf{p}$.

Therefore, by solving this semidefinite program we check whether $A(p)$, $p \in \mathbf{p}$, is weakly positive semidefinite. $\square$

Strong Positive Definiteness

Theorem (The following are equivalent)

(1) *$A(p)$, $p \in \mathbf{p}$, is strongly positive definite,*

(2) *$A(p)$ is positive definite for each p such that $p_k \in \{\underline{p}_k, \overline{p}_k\}$ $\forall k$,*

(3) *$x^T A(p^c)x - \sum_{k=1}^{K} |x^T A^{(k)} x| p_k^\Delta > 0$ for each $0 \neq x \in \mathbb{R}^n$.*

Theorem

(1) *If $A^{(i)}$ is positive semidefinite for some i, then we can fix $p_i := \underline{p}_i$ for checking strong positive definiteness.*

(2) *If $A^{(i)}$ is negative semidefinite for some i, then we can fix $p_i := \overline{p}_i$ for checking strong positive definiteness.*

Theorem (Sufficient Condition)

For each $k = 1, \ldots, K$, split $A^{(k)} = A_1^{(k)} - A_2^{(k)}$ such that both $A_1^{(k)}, A_2^{(k)}$ are positive semidefinite. Then $A(p)$, $p \in \mathbf{p}$, is strongly positive definite if

$$\sum_{k=1}^{K} \left(A_1^{(k)} \underline{p}_k - A_2^{(k)} \overline{p}_k \right) \text{ is positive definite.}$$

Strong Positive Definiteness and Regularity

Definition

$A(p)$, $p \in \mathbf{p}$, is called *regular* if $A(p)$ is nonsingular for each $p \in \mathbf{p}$.

Theorem

The parametric matrix $A(p)$, $p \in \mathbf{p}$, is strongly positive definite if and only if $A(p)$ is positive definite for some $p \in \mathbf{p}$ and $A(p)$, $p \in \mathbf{p}$, is regular.

Beeck sufficient regularity criterion

$A(p)$, $p \in \mathbf{p}$, is regular if

$$\rho(M^{\Delta}) < 1,$$

where

$$\mathbf{M} := \sum_{k=1}^{K} \left(CA^{(k)}\right) \mathbf{p}_k,$$

and $C = A(p^c)^{-1}$ is the preconditioner.

Both sufficient conditions for strong positive definiteness are incomparable.

Application in Convexity Testing

Consider a class of functions

$$f(x) = \sum_{\ell=1}^{L} c_\ell x_{i_\ell} x_{j_\ell} x_{k_\ell},$$

where $i_\ell, j_\ell, k_\ell \in \{0, \ldots, n\}$ are not necessarily mutually different, and $x_0 = 1$.

Problem

Check for convexity of $f(x)$ on $\mathbf{x} \in \mathbb{IR}^n$.

- The Hessian matrix has directly a linear parametric form.
- Each entry of the Hessian of $f(x)$ is a linear function with respect to $x \in \mathbb{R}^n$.
- The variables x play the role of the parameters p, and their domain $\mathbf{x}$ works as $\mathbf{p}$.

Application in Convexity Testing – Example

Example

Check convexity of

$$f(x, y, z) = x^3 + 2x^2y - xyz + 3yz^2 + 5y^2,$$

on $x \in \mathbf{x} = [2, 3]$, $y \in \mathbf{y} = [1, 2]$ and $z \in \mathbf{z} = [0, 1]$. The Hessian of f reads

$$\nabla^2 f(x, y, z) = \begin{pmatrix} 6x + 4y & 4x - z & -y \\ 4x - z & 10 & -x + 6z \\ -y & -x + 6z & 6y \end{pmatrix}.$$

Relaxation leads to

$$\nabla^2 f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \subseteq \begin{pmatrix} [16, 26] & [7, 12] & -[1, 2] \\ [7, 12] & 10 & [-3, 4] \\ -[1, 2] & [-3, 4] & [6, 12] \end{pmatrix},$$

which is not strongly positive semidefinite.

Nevertheless, the sufficient conditions for the parametric Hessian enclosure succeed in proving convexity!

Conclusion

- Extension of characterization of positive (semi)definiteness of interval matrices to parametric forms.
- Surprisingly, finite reduction is possible.
- Even more surprisingly, complexity needn't be worse (from 2^n to 2^K).

References

M. Hladík, D. Daney, and E. Tsigaridas.
Bounds on real eigenvalues and singular values of interval matrices.
SIAM J. Matrix Anal. Appl., 31(4):2116–2129, 2010.

M. Hladík, D. Daney, and E. P. Tsigaridas.
Characterizing and approximating eigenvalue sets of symmetric interval matrices.
Comput. Math. Appl., 62(8):3152–3163, 2011.

L. Jaulin and D. Henrion.
Contracting optimally an interval matrix without loosing any positive semi-definite matrix is a tractable problem.
Reliab. Comput., 11(1):1–17, 2005.

J. Rohn.
Positive definiteness and stability of interval matrices.
SIAM J. Matrix Anal. Appl., 15(1):175–184, 1994.

J. Rohn.
A handbook of results on interval linear problems.
Tech. Rep. 1163, Acad. of Sci. of the Czech Republic, Prague, 2012.