Polyhedral Relaxations for Constraint Satisfaction Problems

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Problem formulation

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\}.$$

The midpoint and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Constraint programming problem

Enclose the set $\mathcal{S}$ described by

$$\begin{aligned} f_i(x_1, \dots, x_n) &= 0, \quad i = 1, \dots, m, \qquad && (\, f(x) = 0\,) \\ g_j(x_1, \dots, x_n) &\le 0, \quad j = 1, \dots, \ell, \qquad && (\, g(x) \le 0\,) \end{aligned}$$

on a box $\mathbf{x}$.

Linearization

Our approach

- linearize constraints,
- compute new bounds and iterate.

Example

$\mathbf{x}$

Linearization

Interval linearization

Let $x^0 \in \mathbf{x}$. Suppose that a function $h : \mathbb{R}^n \mapsto \mathbb{R}^s$ satisfies

$$h(x) \subseteq S_h(\mathbf{x}, x^0)(x - x^0) + h(x^0), \quad \forall x \in \mathbf{x}$$

for a suitable interval-valued function $S_h : \mathbb{IR}^n \times \mathbb{R}^n \mapsto \mathbb{IR}^{s \times n}$.

Techniques

- mean value form
- slopes
- special structure analysis (McCorming-like linearizations ...)

Linearization

Interval linear programming formulation

Now, the set $\mathcal{S}$ is enclosed by

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \le 0,$$

for some interval matrices **A** and **B**.

What remains to do

- Solve the interval linear program
- choose $x^0 \in \mathbf{x}$

Vertex selection of x^0

Case $x^0 := \underline{x}$

Let $x^0 := \underline{x}$. Since $x - \underline{x}$ is non-negative, the solution set to

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \leq 0,$$

is described by

$$\underline{A}x \leq \underline{A}\,\underline{x} - f(\underline{x}), \quad \overline{A}x \geq \overline{A}\underline{x} - f(\underline{x}),$$
$$\underline{B}x \leq \underline{B}\,\underline{x} - g(\underline{x}).$$

- Similarly if x^0 is any other vertex of $\mathbf{x}$
- Araya, Trombettoni & Neveu (2012) recommend two opposite corners

Non-vertex selection of x^0

General case

Let $x^0 \in \mathbf{x}$. The solution set to

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \leq 0,$$

is described by

$$|A_c(x - x^0) + f(x^0)| \leq A_\Delta |x - x^0|,$$
$$B_c(x - x^0) \leq B_\Delta |x - x^0| - g(x^0).$$

- Non-linear description due to the absolute values.
- How to get rid of them?

Non-vertex selection of x^0

Solution

Linearize the absolute values.

Theorem (Beaumont, 1998)

For every $y \in \mathbf{y} \subset \mathbb{R}$ with $\underline{y} < \overline{y}$ one has

$$|y| \leq \alpha y + \beta, \qquad (*)$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \text{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \geq 0$ or $\overline{y} \leq 0$ then $(*)$ holds as equation.

Convex case

Proposition

Let $x^0 \in \mathbf{x}$. Suppose that $\mathbf{A}$ and $\mathbf{B}$ do not depend on a selection of x^0.

1. *If $f_i(x)$ are convex, then the half of the linearized inequalities is a consequence of the corresponding inequalities derived by vertices of $\mathbf{x}$.*
2. *If $f_i(x)$ are concave, then the second half of the linearized inequalities is a consequence of the corresponding inequalities derived by vertices of $\mathbf{x}$.*
3. *If $g_j(x)$ are convex, then the linearized inequality is a consequence of the corresponding inequalities derived by vertices of $\mathbf{x}$.*

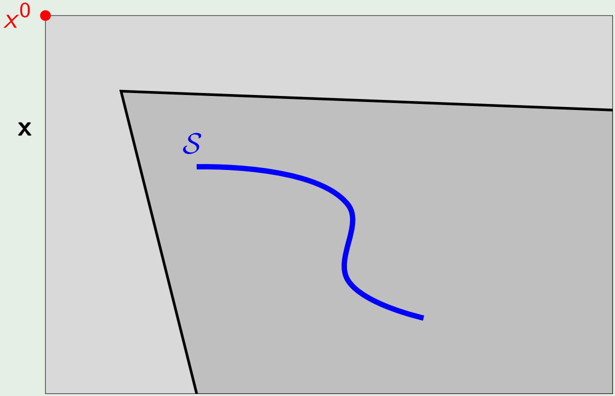
Consequences

- For nice functions (linear, convex), non-vertex selection of x^0 makes no progress
- Non-vertex selection of x^0 is more useful more non-convex are f, g

Example

Example

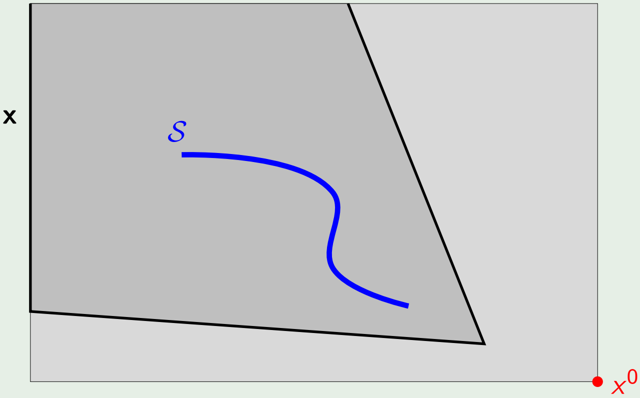
Typical situation when choosing x^0 to be vertex:



Example

Example

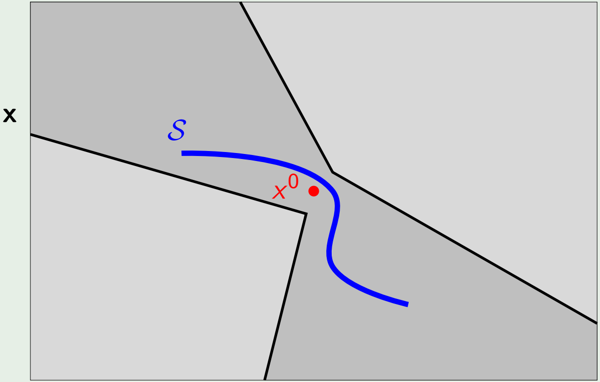
Typical situation when choosing x^0 to be the opposite vertex:



Example

Example

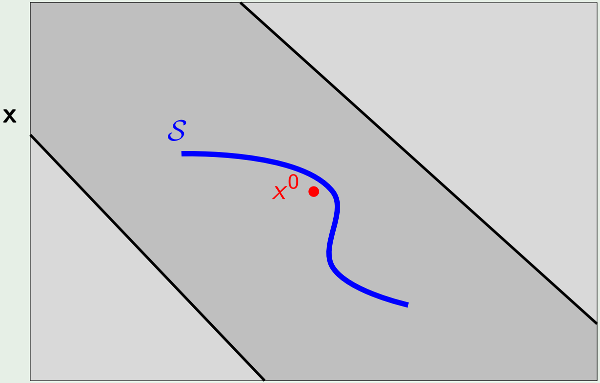
Typical situation when choosing $x^0 = x_c$:



Example

Example

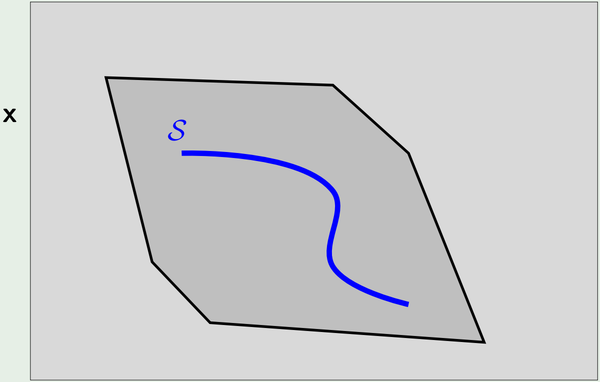
Typical situation when choosing $x^0 = x_c$ (after linearization):



Example

Example

Typical situation when choosing all of them:

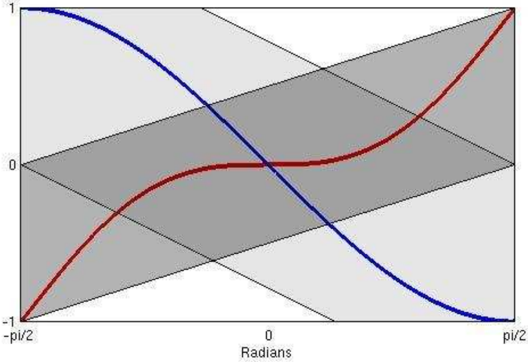


Example II.

Constraints:

$$\pi^2 y - 4x^2 \sin x = 0, \quad y - \cos\left(x + \tfrac{\pi}{2}\right) = 0, \quad x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \ y \in [-1, 1].$$

Center: $x^0 = (0, 0)$

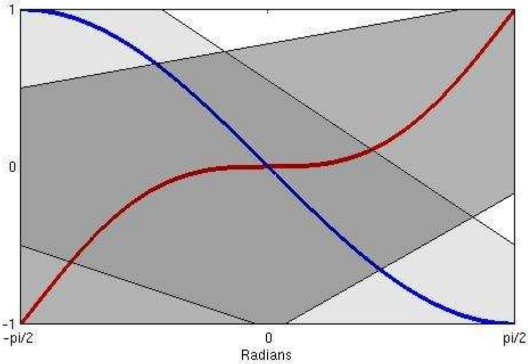


Example II.

Constraints:

$$\pi^2 y - 4x^2 \sin x = 0, \quad y - \cos\left(x + \tfrac{\pi}{2}\right) = 0, \qquad x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \ y \in [-1, 1].$$

Center: $x^0 = (\frac{\pi}{6}, 0)$



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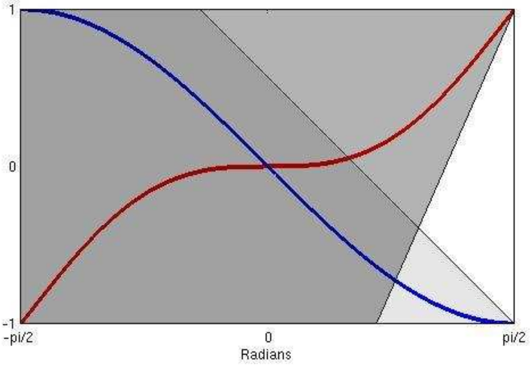

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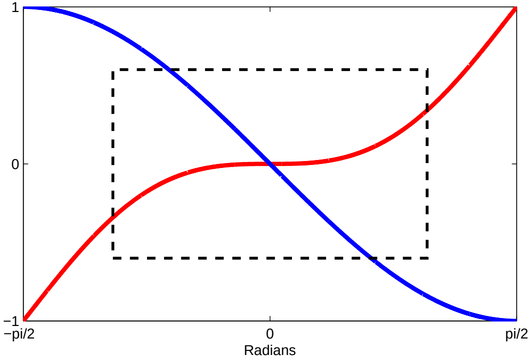
$$\pi^2 y - 4x^2 \sin x = 0, \quad y - \cos\left(x + \tfrac{\pi}{2}\right) = 0, \qquad x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \ y \in [-1, 1].$$

Contraction for centers $x^0 = (0,0), (\frac{\pi}{2}, 0), (-\frac{\pi}{2}, 0)$



Comparison to Parallel Linearization

Suppose that $h : \mathbb{R}^n \mapsto \mathbb{R}^s$ has the following interval linear enclosure on $\mathbf{x}$

$$h(x) \subseteq \mathbf{A}(x - x^0) + h(x^0), \quad \forall x \in \mathbf{x}$$

for a suitable interval matrix $\mathbf{A}$ and $x^0 \in \mathbf{x}$.

Theorem (Jaulin, 2001)

For any $A \in \mathbf{A}$ we have

$$h(x) \geq A(x - x^0) + h(x^0) + \underline{(\mathbf{A} - A)(\mathbf{x} - x^0)},$$
$$h(x) \leq A(x - x^0) + h(x^0) + \overline{(\mathbf{A} - A)(\mathbf{x} - x^0)}.$$

Theorem

For any selection of $x^0 \in \mathbf{x}$ and $A \in \mathbf{A}$, the interval linear programming approach yields always as tight enclosures as the parallel linearization.

Summary, conclusion and future work

At each iteration

- choose two opposite corners and the midpoint x_c
- we get a system of $3(2m + \ell)$ inequalities with respect to n variables
- solve $2n$ linear programs to have a new box $\mathbf{x}' \subseteq \mathbf{x}$

Properties

- Runs in polynomial time, applicable for larger dimensions.

Future work

- choice of x^0: optima of the linear programs?
  optima of underestimators (in global optimization)
  what number?

References

M. Hladík and J. Horáček.
Interval linear programming techniques in constraint programming and global optimization.
submitted to LNCS, 2013.