Optimization with uncertain, inexact or unstable data: Linear programming and the interval approach

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Part I. Introduction

- What is optimization?
- Applications in Operations Research, Economics, Statistics, Game Theory, ...
- Imprecision of data
- Formalization of Interval Linear Programming

What is optimization?

- **In mathematics:** optimization is a theory on maximization or minimization of functions defined on special sets.
  - We are given an **objective function** $f : \mathbb{R}^n \to \mathbb{R}$ which is to be maximized/minimized over a given region $\Omega \subseteq \mathbb{R}^n$, called **feasible space**.
- **In applications:** we are always maximizing or minimizing something, for example:
  - **Microeconomics:** we want to maximize profit of a firm or minimize its costs under limited resources;
  - **Statistics:** we want to maximize likelihood or minimize residual error (e.g. residual sum of squares);
  - **Experimental Design:** we want to maximize measurement precision in an experiment under restricted possibilities of available laboratory equipment;
  - **Portfolio Theory:** we want to maximize return of an investment under budget constraints and regulatory constraints;
  - **Operations Research:** we want to minimize the length of the path of the Traveling Salesman;
  - many more applications in engineering, physics, chemistry etc.

A basic example: Nutrition (Diet) Problem

In practice, this problem is solved e.g. by producers of dog food, often on a daily basis.

- A producer of dog food processes leftovers from butchers, slaughterhouses and meat-processing plants.
- The producer must combine the available raw materials to achieve the declared nutrition content, e.g. enough proteins, enough calories, not too much salt, not too much fat etc.
- The producer does not care about what the ingredients exactly are:
  - (s)he simply buys anything from which it is possible to combine the declared nutrition levels, as cheaply as possible,
  - (s)he mixes and boils the raw materials, getting a homogenous tasteless mesh,
  - (s)he adds meat perfume,
  - (s)he fills the 'product' into cans,
  - (s)he adds one piece of real meat just under the cover of each can (for a better visual effect),
  - (s)he spends plenty of money on marketing to be able to sell this stuff.

A basic example: Nutrition Problem (continued)

For example assume that a meat processing plant offers two kinds of leftovers:

- x_1 = a salami called *Gothai*,
- x_2 = mechanically separated meat.

Remark. *Gothai* is a legendary Czech salami; to illustrate, roasted *Gothai* with vinegar and raw onion is the most beloved dish of the current president of the Czech republic.

Now we can summarize data for our problem: we know the contents of proteins, fat, and salt in each of the two ingredients x_1, x_2, and we know their prices per ton.

Our goal. We want to mix them to obtain dog food containing at least a declared level of proteins, at most a declared level of fat and at most a declared level of salt. Furthermore, we want to minimize costs.

A basic example: Nutrition Problem (continued)

We get the following optimization problem:

	Gothai		sep.meat		demand
minimize	$c_1 x_1$	$+$	$c_2 x_2$		
subject to:					
proteins	$a_{11} x_1$	$+$	$a_{12} x_2$	$\geqslant$	b_1
fat	$a_{21} x_1$	$+$	$a_{22} x_2$	$\leqslant$	b_2
salt	$a_{31} x_1$	$+$	$a_{32} x_2$	$\leqslant$	b_3
	x_1			$\geqslant$	0
			x_2	$\geqslant$	0

Data of the optimization problem are denoted in red:

- a_{ij} denote the contents of proteins, fat, salt in one tone of Gothai and separated meat,
- b_i denote the demands,
- c_j denote the prices per ton of Gothai and separated meat.

From a crisp to an interval setting

Remark. This was an example of a linear programming problem: both the cost function and the constraints were linear. In this lecture we will deal only with linear programming problems; however, many of the ideas can be translated to the nonlinear setup as well.

What often happens in practice: the data a_{ij}, b_i, c_j are not reliable — they are subject to imprecision, instability or changes.

- Say that it is declared that Gothai contains 30% of meat and 45% of fat. But when the delivery arrives, the producer's laboratory finds out that the true contents is 20% and 50%, respectively.
- Prices might be subject to intraday changes — the true delivery prices c_1, c_2 might be different from the declared prices. (This happens, for example, in metal markets, where instantaneous prices are driven by the London Metal Exchange.)
- Demands might be subject to changes: should the product be sold in Switzerland, the quality must be higher (say, with a higher demand for the contents of meat in the dog food), but for a delivery into the Czech republic lower quality is sufficient. (At the moment the producer does not know which case will occur.)

From a crisp to an interval setting (continued)

How can we incorporate imprecision into the model? There are (at least) three approaches studied in literature:

- data a_{ij}, b_i, c_j are understood as random variables $\longrightarrow$ stochastic programming;
- data a_{ij}, b_i, c_j are understood as fuzzy numbers $\longrightarrow$ fuzzy linear programming;
- data a_{ij}, b_i, c_j are understood as intervals $\longrightarrow$ interval linear programming.

We will study the last case: now, the crisp data a_{ij}, b_i, c_j are replaced by closed intervals $[\underline{a}_{ij}, \overline{a}_{ij}]$, $[\underline{b}_i, \overline{b}_i]$, $[\underline{c}_j, \overline{c}_j]$ of possible values.

The dog-food example

So we get an interval version of the dog-food linear program:

	Gothai		sep.meat		demand
minimize	$[\underline{c}_1, \overline{c}_1]x_1$	$+$	$[\underline{c}_2, \overline{c}_2]x_2$		
subject to:					
proteins	$[\underline{a}_{11}, \overline{a}_{11}]x_1$	$+$	$[\underline{a}_{12}, \overline{a}_{12}]x_2$	$\geqslant$	$[\underline{b}_1, \overline{b}_1]$
fat	$[\underline{a}_{21}, \overline{a}_{21}]x_1$	$+$	$[\underline{a}_{22}, \overline{a}_{22}]x_2$	$\leqslant$	$[\underline{b}_2, \overline{b}_2]$
salt	$[\underline{a}_{31}, \overline{a}_{31}]x_1$	$+$	$[\underline{a}_{32}, \overline{a}_{32}]x_2$	$\leqslant$	$[\underline{b}_3, \overline{b}_3]$
	x_1			$\geqslant$	0
			x_2	$\geqslant$	0

Natural questions: How to translate notions from traditional linear programming to the interval setting? What is a *feasible solution*? What is an *optimal solution*? What does *unboundedness* mean? What does *infeasibility* mean? Etc.

Motivation for interval data

Interval data are very frequent. Sources of intervals:

- **Measurement errors and rounding errors.** Results of measurements are often expressed in the form of $v \pm \Delta v$.
- **Nonconstant constants.** Many physical and chemical constants are *not constant*.
- **Unobservable data.** Example: inflation expectations.
- **Discretization.** Continuous variables are discretized into a finite set of values – e.g., time is split into time slots (days, years, ...). For example, stock indices.
- **Categorization or Incomplete information.** Input data is incomplete due to protection of privacy or just due to lack of knowledge. For instance, salary, age, etc.

Part II. Theory and Applications

- General formulation
- Weak and strong properties
- Optimal value function
- Inverse optimization
- Interesting research problems

Notation

Notation.

- An $(m \times n)$-interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is the family of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} : \underline{A} \leqslant A \leqslant \overline{A}\},$$

where the inequality "$\leqslant$" is understood entrywise.

- Interval matrices are denoted in boldface: $\mathbf{A}, \mathbf{B}, \dots$.
- The center and radius matrix of an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ are defined, respectively, as

$$A^c = \tfrac{1}{2}(\overline{A} + \underline{A}), \quad A^\Delta = \tfrac{1}{2}(\overline{A} - \underline{A}).$$

- The notation for interval vectors is similar.

An interval linear program with data $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is defined as a family of linear programs

- $\min c^{\mathrm{T}}x$ s.t. $Ax \leqslant b$, $\quad A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, or
- $\min c^{\mathrm{T}}x$ s.t. $Ax = b$, $x \geqslant 0$, $\quad A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, or
- $\min c^{\mathrm{T}}x$ s.t. $Ax \leqslant b$, $x \geqslant 0$, $\quad A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Each LP in the family is called scenario.

We simply denote the families as

- $\min \mathbf{c}^{\mathrm{T}}x$ s.t. $\mathbf{A}x \leqslant \mathbf{b}$,
- $\min \mathbf{c}^{\mathrm{T}}x$ s.t. $\mathbf{A}x = \mathbf{b}$, $x \geqslant 0$,
- $\min \mathbf{c}^{\mathrm{T}}x$ s.t. $\mathbf{A}x \leqslant \mathbf{b}$, $x \geqslant 0$,

respectively.

Interval linear programs: weak/strong properties

- **Strong property:** *every* LP in the family has it.
- **Weak property:** *some* LP in the family has it.

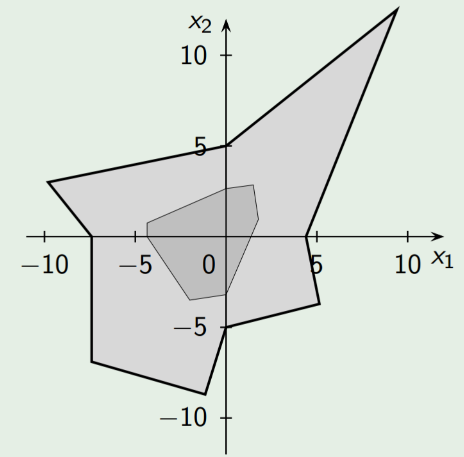
Examples:

- strong/weak feasibility,
- strong/weak unboundedness,
- strong/weak optimality (meaning: is a given basis optimal for every/some LP in the family?),
- strongly/weakly feasible region.

Remark. Strong property shows robustness of the model w.r.t the property.

Interval linear programs: feasible region

- **Strongly feasible region:** set of points which are feasible for *every* LP in the family.
- **Weakly feasible region:** set of points which are feasible for *some* LP in the family.



$$\begin{pmatrix} -[2,5] & -[7,11] \\ [1,13] & -[4,6] \\ [5,8] & [-2,1] \\ -[1,4] & [5,9] \\ -[5,6] & -[0,4] \end{pmatrix} x \leq \begin{pmatrix} [61,63] \\ [19,20] \\ [15,22] \\ [24,25] \\ [26,37] \end{pmatrix}$$

Interval linear programs: do formulations differ?

Fact. In traditional linear programming it is well-known that the three mentioned formats

(a) $\min c^{\mathrm{T}}x$ s.t. $Ax \leqslant b$,

(b) $\min c^{\mathrm{T}}x$ s.t. $Ax = b,\ x \geqslant 0$,

(c) $\min c^{\mathrm{T}}x$ s.t. $Ax \leqslant b,\ x \geqslant 0$

are equivalent, meaning that an LP in one format is easily (in polynomial time) reducible onto the other formats.

For example, the (b)-form LP $\min c^{\mathrm{T}}x$ s.t. $Ax = b,\ x \geqslant 0$ can be easily written in the (a)-form as

$$\min c^{\mathrm{T}}x \ \text{ s.t. } \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \leqslant \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}.$$

Question. Does something similar hold for interval LPs?

Answer. No, at least from the computational perspective.

Complexity of strong/weak properties

format:	$\mathbf{A}x = \mathbf{b}, x \geqslant 0$	$\mathbf{A}x \leqslant \mathbf{b}$	$\mathbf{A}x \leqslant \mathbf{b}, x \geqslant 0$
strong feasibility	co-NP-hard	poly-time	poly-time
weak feasibility	poly-time	NP-hard	poly-time
strong unboundedness	co-NP-hard	poly-time	poly-time
weak unboundedness	open problem	open problem	poly-time
strong optimality	co-NP-hard	co-NP-hard	poly-time
weak optimality	open problem	open problem	open problem

To recall:

- (co-)NP-hardness is often interpreted as "only exponential-time or worse algorithms for the problem exist", or "the problem is computationally intractable" (it is as hard as general integer programming, Traveling Salesman etc., or even harder).
- poly-time ("polynomial computation time") is often interpreted as "the problem is computationally tractable even for large instances".

Optimal value range and the catastrophic scenario

The optimal value function is defined as

$$f(A, b, c) = \min\{c^{\mathrm{T}}x : Ax = b, x \geqslant 0\}$$

on $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$. (For other LP formats the definition is analogous.)

Natural questions:

- How to determine $\overline{f} := \sup\{f(A, b, c) : (A, b, c) \in (\mathbf{A}, \mathbf{b}, \mathbf{c})\}$?
- How to determine $(A^U, b^U, c^U) \in (\mathbf{A}, \mathbf{b}, \mathbf{c})$ s.t. $f(A^U, b^U, c^U) = \overline{f}$?

Remark. When the objective function $c^{\mathrm{T}}x$ measures costs, then $(A^U, b^U, c^U, \overline{f})$ is the catastrophic scenario: it tells us what can happen in the worst case, when the cost function attains its highest possible value.

Optimal value range and the catastrophic scenario (contd.)

Also the opposite extreme ("best case") is interesting:

- How to determine $\underline{f} := \inf\{f(A,b,c) : (A,b,c) \in (\mathbf{A},\mathbf{b},\mathbf{c})\}$?
- How to determine $(A_L, b_L, c_L) \in (\mathbf{A},\mathbf{b},\mathbf{c})$ such that $f(A_L, b_L, c_L) = \underline{f}$?

Definition. The pair of (possibly infinite) numbers $\underline{f}, \overline{f}$ is called optimal value range.

Questions.

- Can we easily determine the range $\underline{f}, \overline{f}$?
- Can we easily find the extreme scenarios (A^U, b^U, c^U), (A_L, b_L, c_L)?
- Not surprisingly, the answer depends on the format of LP.

Optimal value range and the catastrophic scenario (contd.)

We can extend our table:

format:	$\mathbf{A}x = \mathbf{b}, x \geqslant 0$	$\mathbf{A}x \leqslant \mathbf{b}$	$\mathbf{A}x \leqslant \mathbf{b}, x \geqslant 0$
strong feasibility	co-NP-hard	poly-time	poly-time
weak feasibility	poly-time	NP-hard	poly-time
strong unboundedness	co-NP-hard	poly-time	poly-time
weak unboundedness	open problem	open problem	poly-time
strong optimality	co-NP-hard	co-NP-hard	poly-time
weak optimality	open problem	open problem	open problem
computing $\overline{f}$	NP-hard	poly-time	poly-time
computing $\underline{f}$	poly-time	NP-hard	poly-time

Optimal value range — algorithms

Vajda's Theorem (1961). For the format $\mathbf{A}x \leqslant \mathbf{b}, x \geqslant 0$ we have

- $\underline{f} = \min\{\underline{c}^{\mathrm{T}}x : \underline{A}x \leqslant \overline{b}, x \geqslant 0\}$,
- $\overline{f} = \min\{\overline{c}^{\mathrm{T}}x : \overline{A}x \leqslant \underline{b}, x \geqslant 0\}$.

Remark. In this case, computation of the optimal value range is reducible to traditional LP.

Rohn's Theorem (2006). For the format $\mathbf{A}x = \mathbf{b}, x \geqslant 0$ we have

- $\underline{f} = \min\{\underline{c}^{\mathrm{T}}x : \underline{A}x \leqslant \overline{b}, \overline{A}x \geqslant \underline{b}, x \geqslant 0\}$,
- $\overline{f} = \max\{f(A^c - \mathrm{diag}(s)A^{\Delta}, b_c + \mathrm{diag}(s)b^{\Delta}, \overline{c}) : s \in \{\pm 1\}^n\}$,

where A^c denotes the center matrix, A^{Δ} denotes the radius matrix and $\mathrm{diag}(\xi_1, \ldots, \xi_n)$ denotes the diagonal matrix with diagonal elements $\xi_1, \ldots, \xi_n$.

Remark. In this case, $\underline{f}$ is computed by one traditional LP; but for $\overline{f}$, we need to solve 2^n traditional LPs. (The NP-hardness results show that we can expect nothing much better.)

Do we need to investigate different form of interval LP separately?

- Yes. Because some forms are tractable, while the others are not.
- No. We can put it in a general setting.

The general form of interval LP:

$$\max \mathbf{c}^{\mathrm{T}} x + \mathbf{d}^{\mathrm{T}} y \ \text{ s.t. } \ \mathbf{A}x + \mathbf{B}y = \mathbf{b}, \ \mathbf{C}x + \mathbf{D}y \leqslant \mathbf{g}, \ x \geqslant 0.$$

Remarks.

- Each interval LP can be expressed in this form.
- Algorithms for computing $\underline{f}$ and $\overline{f}$ are known, but not necessarily efficient (not necessarily poly-time).

Optimal value function — continuity

Another natural question. Is the optimal value function $f(A, b, c) = \min\{c^{\mathrm{T}}x : Ax = b, x \geqslant 0\}$ continuous on $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$?

Answer. In general: no. For example, it can happen that there is a number f_0 s.t. $\underline{f} < f_0 < \overline{f}$ for which no $(A_0, b_0, c_0) \in (\mathbf{A}, \mathbf{b}, \mathbf{c})$ satisfies $f(A_0, b_0, c_0) = f_0$.

But under some additional conditions, the answer is positive:

Theorem (Wets, 1985; Mostafaee, Hladík, Černý, 2013). Assume that for every $(A, b, c) \in (\mathbf{A}, \mathbf{b}, \mathbf{c})$ the following conditions hold:

- $\{x \in \mathbb{R}^n : Ax = 0, x \geqslant 0, c^{\mathrm{T}}x \leqslant 0\} = \{0\}$,
- $\{y \in \mathbb{R}^m : A^{\mathrm{T}}y \leqslant 0, b^{\mathrm{T}}y \geqslant 0\} = \{0\}$.

Then both values $\underline{f}$ and $\overline{f}$ are finite and $f(A, b, c)$ is continuous on $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$.

Application: inverse optimization

Inverse optimization is a task to *design* a linear program with a prescribed optimal value, when its coefficients can be chosen from given intervals.

- Example 1: design a network with a prescribed maximal flow, when capacities of edges can be chosen from given intervals.
- Example 2: find a payoff matrix for a matrix game with a prescribed value of the game, when payoffs can be chosen from given intervals.

We will elaborate on Example 2 in more detail.

Given a payoff matrix A for a matrix game, it is well-known that the Nash mixed strategy for the first player can be found via the linear program

$$\max \gamma \quad \text{s.t.} \quad Ax \geqslant \gamma e, \quad e^{\mathrm{T}} x = 1, \quad x \geqslant 0, \tag{1}$$

where $e = (1, \ldots, 1)^{\mathrm{T}}$.

Observations. Let $\mathbf{A}$ be given.

- The LP (1) satisfies the assumptions of Wets-Mostafaee-Hladík-Černý's Theorem and hence f is continuous.
- We have $\underline{f} = \max\{\gamma : \underline{A}x \geqslant \gamma e, e^{\mathrm{T}} x = 1, x \geqslant 0\}$.
- We have $\overline{f} = \max\{\gamma : \overline{A}x \geqslant \gamma e, e^{\mathrm{T}} x = 1, x \geqslant 0\}$.

So, in this case, we know both extreme scenarios.

Inverse optimization: designing a matrix game

Efficient method for solving the inverse optimization problem — designing a payoff matrix for a matrix game. Given an interval matrix $\mathbf{A}$ of admissible payoffs and a prescribed value $f_0 \in (\underline{f}, \overline{f})$ for the desired value of the game, we can define the function

$$\Theta(\lambda) = \max\{\gamma : ((1-\lambda)\underline{A} + \lambda\overline{A})x \geqslant \gamma e, e^{\mathrm{T}} x = 1, x \geqslant 0\} - f_0.$$

By continuity, Θ has a zero point (using Bolzano's Intermediate Value Theorem). This point can be found using the Binary Search technique, simply iterating over $\lambda \in [0, 1]$.

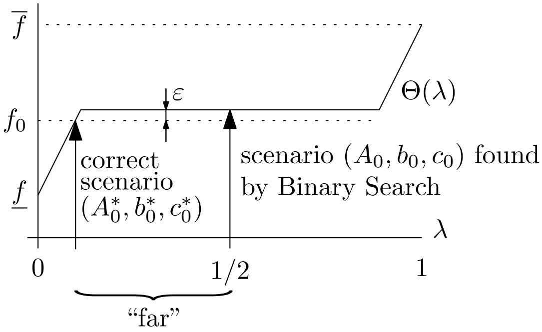
Applications.

- When a casino wants to introduce a matrix game to be played, it is desirable to setup the payoff matrix to achieve the desired game value $-\frac{1}{37}$ (i.e., the same value as roulette has).
- When a casino wants to introduce a matrix game to be played, it is desirable to determine a fee for playing the game which would lead to the desired game value $-\frac{1}{37}$.

Some interesting research problems

- Weaken the sufficient conditions for continuity of the optimal value function f from Wets-Mostafaee-Hladík-Černý's Theorem. (There are many examples when the function is continuous but the conditions are not satisfied.)
- The solution to the inverse optimization problem need not be unique. So, define an ordering "this solution is better than that one" and design an algorithm for finding the "better" solutions.
- The Binary Search technique finds the solution only approximately, with an arbitrary given precision $\varepsilon > 0$. Find another method which will find a solution exactly.
- Find an estimate on the number of iterations of the Binary Search technique necessary for achieving a given precision $\varepsilon > 0$. (This requires an analysis of how "wildly" the optimal value function f can behave.)

Some interesting research problems (contd.)



Part III. Dependency problem

Though the three mentioned formats

(a) $\min c^{\mathrm{T}}x$ s.t. $\mathbf{A}x \leqslant \mathbf{b}$,

(b) $\min c^{\mathrm{T}}x$ s.t. $\mathbf{A}x = \mathbf{b},\ x \geqslant 0$,

(c) $\min c^{\mathrm{T}}x$ s.t. $\mathbf{A}x \leqslant \mathbf{b},\ x \geqslant 0$

look very general, in fact they require a (sometimes) restrictive condition: *data of the linear program are independent*. Said roughly, it means that data can occur only once in the formulation of an LP.

To illustrate a case when this condition is violated: consider the generic diet problem, where we have *both lower and upper bounds on the content of fat*.

Dependency problem (Example)

	Gothai		sep.meat		demand
minimize	$c_1 x_1$	$+$	$c_2 x_2$		
subject to:					
proteins	$a_{11} x_1$	$+$	$a_{12} x_2$	$\geqslant$	b_1
fat (upper bound)	$a_{21} x_1$	$+$	$a_{22} x_2$	$\leqslant$	b_2
fat (lower bound)	$a_{31} x_1$	$+$	$a_{32} x_2$	$\geqslant$	b_3
salt	$a_{41} x_1$	$+$	$a_{42} x_2$	$\leqslant$	b_4
	x_1			$\geqslant$	0
			x_2	$\geqslant$	0

Here, we must ensure that it always holds

$$a_{21} = a_{31} \quad \text{and} \quad a_{22} = a_{32}.$$

The corresponding left-hand coefficients in 2nd and 3rd constraint must *always be the same*. We say that they are dependent.

Dependency problem (contd.)

When turning into the interval setting, it is not appropriate to consider the entire family of linear programming problems

$$\{\min\{c^{\mathrm{T}}x : Ax \leqslant b, x \geqslant 0\} \mid A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}, \tag{2}$$

but only a subfamily

$$\{\min\{c^{\mathrm{T}}x : Ax \leqslant b, x \geqslant 0\} \mid A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c} \ \text{ s.t. } \ a_{21} = a_{31} \text{ and } a_{22} = a_{32}\}. \tag{3}$$

However, by our definition, (3) is not an interval linear program!

The interval linear program (2) is called relaxation of (3). The conditions $a_{21} = a_{31}$ and $a_{22} = a_{32}$ are called dependency conditions.

Dependency problem (contd.)

Why are dependencies interesting?

- We have almost no theory on interval LPs with dependencies. So, their investigation is a challenging research problem. (Any ideas are welcome.)
- In fact, almost the only thing we can do is working with the relaxation.
- But this brings a variety of problems: $\underline{f}, \overline{f}$ are under/overestimated and many conditions are weakened. (For example, when the relaxation is strongly bounded, then also the original ILP with dependencies is strongly bounded; but this does not hold vice versa.)
- The dependency problem is often serious:
  - it suffices to imagine that we need to use an LP constraint of the form $b \leqslant a^{\mathrm{T}}x \leqslant b'$, or
  - when we want to rewrite an equality $a^{\mathrm{T}}x = b$ as a pair of inequalities $a^{\mathrm{T}}x \leqslant b$, $a^{\mathrm{T}}x \geqslant b$.

To conclude: we need help! SOS!

Part IV. A brief summary of further results and open problems

Interesting special case: Basis stability

Definition. The interval LP

$$\min \mathbf{c}^{\mathrm{T}} x \quad \text{s.t.} \quad \mathbf{A}x = \mathbf{b}, x \geqslant 0 \tag{4}$$

is basis stable iff there exists a basis B which is optimal for every LP in the family (4).

Bad news. Testing basis stability is NP-hard.

Good news.

- Under basis stability we have an efficient procedure for determining the optimal value range:
  - $\overline{f} = \min\{\overline{c}_B^{\mathrm{T}} x : \underline{A}_B x_B \leqslant \overline{b}, \overline{A}_B x_B \geqslant \underline{b}, x_B \geqslant 0\}$,
  - $\underline{f} = \min\{\underline{c}_B^{\mathrm{T}} x : \underline{A}_B x_B \leqslant \overline{b}, \overline{A}_B x_B \geqslant \underline{b}, x_B \geqslant 0\}$.
- Moreover, the set of optimal solutions of LPs in the family (4) is the polyhedron

$$\{x \in \mathbb{R}^n : \underline{A}_B x_B \leqslant \overline{b}, \overline{A}_B x_B \geqslant \underline{b}, x_B \geqslant 0, x_{\text{nonbasic}} = 0\}.$$

Application: linear regression with interval data

Consider the linear regression model

$$y = X\beta + \varepsilon,$$

where data (X, y) are not observable. Instead we have only intervals $(\mathbf{X}, \mathbf{y})$ containing the unobservable values (X, y).

L_1-norm estimator $\widehat{\beta} = \text{argmin}_b \|y - Xb\|_1$ can be written as a linear program:

$$\min e^{\mathrm{T}} w \quad \text{s.t.} \quad Xb - y \leqslant w, \; -Xb + y \leqslant w, \; w \geqslant 0.$$

Replacement of (X, y) by $(\mathbf{X}, \mathbf{y})$ leads to the interval linear program

$$\min e^{\mathrm{T}} w \quad \text{s.t.} \quad \mathbf{X}b - \mathbf{y} \leqslant w, \; -\mathbf{X}b + \mathbf{y} \leqslant w, \; w \geqslant 0.$$

(Note that the dependency problem is present here.)

Application: linear regression with interval data (contd.)

Illustration of basis stability. Consider the estimated "regression line" as a classifier which classifies point into two groups: points *above* the line and *below* the line.

Now: the problem is basis stable iff every choice of data $(X, y) \in (\mathbf{X}, \mathbf{y})$ leads to the same classification.

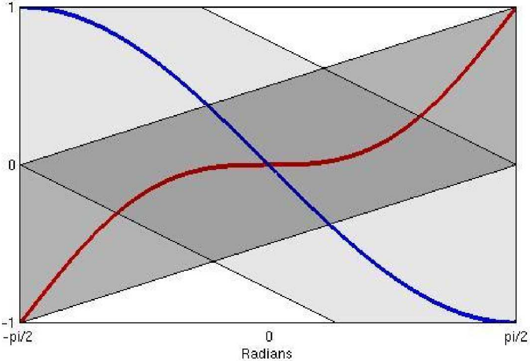
Global optimization problem.

$$\min \varphi(x) \quad \text{s.t.} \quad f(x) = 0,\; g(x) \leqslant 0,\; x \in \mathbf{x}.$$

Linearization of nonlinearities leads to Interval LP.

$$\min \mathbf{c}^T x \quad \text{s.t.} \quad \mathbf{A}x = 0,\; \mathbf{B}x \leqslant 0,\; x \in \mathbf{x}.$$

Illustration. Relaxation of two nonlinear functions:



Interesting open problems

- Find necessary and/or sufficient conditions for
  - weak unboundedness,
  - strong boundedness,
  - weak point optimality (given a point x, is it optimal for some LP in the family?),
  - weak basis optimality (given a basis B, is it optimal for some LP in the family?).
- Find a method which describes the range of the objective value function (i.e., the set $f(\mathbf{A}, \mathbf{b}, \mathbf{c})$).
- In particular, decide whether or not it is an interval.

Further issues

Further issues studied in interval LP:

- approximations of the weak optimal region (= union of optimal solutions of LPs in the family),
- duality in interval LP,
- treatment of dependencies,
- many further topics.

Thank You! And... some further reading for evenings...

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