A shaving method for interval linear systems of equations

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Introduction

Interval matrix

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The midpoint and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Interval linear system

Given $\mathbf{A}$ and $\mathbf{b}$, an interval linear system is a family

$$Ax = b, \quad A \in \mathbf{A},\ b \in \mathbf{b},$$

Its solution set is defined

$$\Sigma := \{x \in \mathbb{R}^n \mid \exists A \in \mathbf{A}\ \exists b \in \mathbf{b} : Ax = b\}.$$

Introduction

Problem formulation

Find a tight interval enclosure $\mathbf{x} \supseteq \Sigma$.

Bad news

The problem of computation (or with prescribed accuracy) the best possible enclosure is NP-hard (Kreinovich and Lakeyev, 1996).

Good news

There are many methods for computing enclosures to Σ:

- Fast methods, but sometimes poor enclosures:
  - Gaussian elimination, Gauss–Seidel or Krawczyk iterations, ε-inflation (Rump, 1994), Hansen–Bliek–Rohn-Ning-Kearfott method (1999)
- Best enclosure, but exponential worst case complexity:
  - Jansson (1997), Rohn (2005)

Our objective

Fill the gap: Polynomial algorithm yielding tight enclosures.

Illustration

Example (sometimes this case . . .)

$$\begin{pmatrix} [6,7] & [2,3] \\ [1,2] & -[4,5] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [6,8] \\ -[7,9] \end{pmatrix}$$

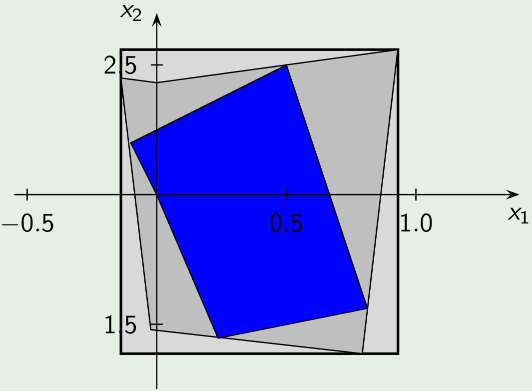
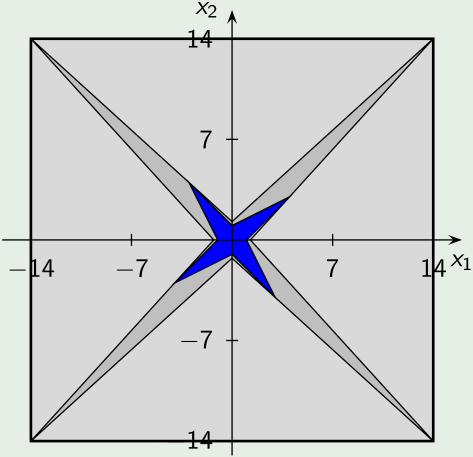


Illustration II.

Example (...and sometimes this case (Barth & Nuding, 1974))

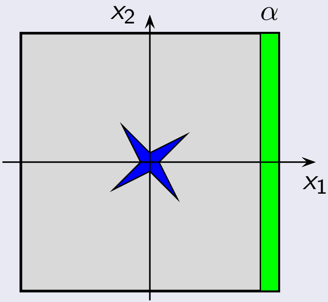
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$



Shaving method

Our idea

Use shaving approach from CSP area.



Form of a slice

Consider a slice $\mathbf{x} = \mathbf{x}(\alpha, i)$ of an initial enclosure $\mathbf{x}^0$ in the form of

$$\mathbf{x} = \begin{cases} \mathbf{x}_j^0 & \text{if } j \neq i, \\ [\overline{x}_j^0 - \alpha, \overline{x}_j^0] & \text{if } j = i, \end{cases}$$

Auxiliaries

Lemma

Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ *and* $\mathbf{x} \in \mathbb{IR}^n$. *Then the linear system*

$$Ax = b, \quad x \in \mathbf{x}$$

has no solution if and only if the linear system

$$A^T w + y - z = 0, \quad b^T w + \overline{x}^T y - \underline{x}^T z = -1, \quad y, z \geq 0 \tag{$*$}$$

is solvable.

Proof.

Consequence of Farkas' lemma. □

Auxiliaries (cont'd)

A sufficient condition for strong solvability of $(*)$

- Let w^*, y^*, z^* be optimal solitions to the linear program

$$\min b_c^T w + \overline{x}^T y - \underline{x}^T z$$
$$\text{subject to } \; A_c^T w + y - z = 0, \; -e \le w \le e, \; y, z \ge 0.$$

- If $y_i^* = 0$, then we fix the variable $y_i = 0$, and the same for z.
- Complete

$$A^T w + y - z = 0, \quad b^T w + \overline{x}^T y - \underline{x}^T z = -1,$$

to the square interval linear system

$$Cv = d, \quad C \in \mathbf{C}.$$

- Let $(\mathbf{v}^1, \mathbf{v}^2)$ be an enclosure of its solution set. If $\underline{v}^2 \ge 0$, then $(*)$ is strongly solvable.

Computing the width of a slice

Our problem

Determine a large value of $\alpha \geq 0$ such that an enclosure $(\mathbf{v}^1, \mathbf{v}^2)$ to

$$(C + \alpha E_{ij})v = d, \quad C \in \mathbf{C},$$

satisfies $\underline{v}^2 \geq 0$.

Solution

Let $\mathbf{v}$ an enclosure to $Cv = d$, $C \in \mathbf{C}$. By the Sherman–Morrison formula for the inverse, we get bounds:

$$\alpha < -1/\underline{C^{-1}_{ji}},$$

$$\alpha \leq \frac{\underline{v}_k}{\overline{\mathbf{v}_j \mathbf{C}^{-1}_{ki}} - \underline{v}_k \underline{C^{-1}_{ji}}}. \quad \forall k \in I : \overline{\mathbf{v}_j \mathbf{C}^{-1}_{ki}} > \underline{v}_k \underline{C^{-1}_{ji}}.$$

Computing the width of a slice (cont'd)

Iterations

The process can be iteration with efficient recomputation of $\mathbf{C}_{*i}^{-1}$.

Remark (Computational complexity)

The total computational time is

$$\mathcal{O}(\mathit{iter} \cdot n \cdot (LP + n^3)),$$

where

- LP is the running time for the linear program
- *iter* is the number of iterations

Example

Example

$$A \in \mathbf{A} = \begin{pmatrix} -[6,7] & [8,10] \\ [5,6] & -[1,3] \end{pmatrix}, \quad b \in \mathbf{b} = \begin{pmatrix} -[10,11] \\ [-1,1] \end{pmatrix}.$$

- The initial enclosure (by the Intlab function `verifylss`)

$$\mathbf{x}^0 = ([-2.1891, 1.0385],\ [-3.2972, 0.1329])^T$$

- Shaving:
  - for $i = 1$: $\alpha_1 = 0.8521$,
  - for $i = 2$: $\alpha_2 = 0.7142$, $\alpha_3 = 0.1669$, $\alpha_4 = 0.0657$,
  - shaving from below inefficient.
- The resulting enclosure

$$\mathbf{x}^1 = ([-2.1891, 0.1864],\ [-3.2972, -0.8139])^T.$$

- the interval hull of Σ is

$$\mathbf{x}^3 = ([-2.1579, 0],\ [-3.2632, -1])^T.$$

Example (cont'd)

Example

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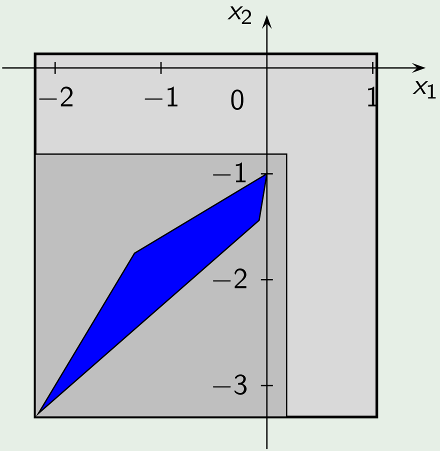

Example II.

Example (Random tests)

- The entries of A_c and b_c randomly in $[-10, 10]$; all radii are equal to $\delta > 0$.
- Results:

n	δ	time	sum	prod	cuts
5	0.5	0.4977	0.6465	0.07751	18.06
10	0.25	0.9941	0.6814	0.02184	45.06
20	0.05	3.136	0.7161	0.00639	87.77
50	0.025	26.65	0.8071	0.03424	281.9
100	0.01	228.5	0.8693	0.01531	946.3

where

$$\mathtt{sum} := \frac{\sum_{i=1}^{n} (x_\Delta^1)_i}{\sum_{i=1}^{n} (x_\Delta^0)_i}, \quad \mathtt{prod} := \frac{\prod_{i=1}^{n} (x_\Delta^1)_i}{\prod_{i=1}^{n} (x_\Delta^0)_i},$$

$\mathbf{x}^0$ is the initial box, and $\mathbf{x}^1$ the computed one.

Conclusion and future work

Conclusion

- Shaving method for solving interval linear equations presented.
- Compromise between time and accuracy:
  polynomial time complexity, but tighter enclosures.

Future work

- Implementation of an efficient parallelization.
- Adaptation to parametric interval linear systems.

Thank you for your attention!