Inverse Linear Optimization with Interval Coefficients

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ICPAM-LAE 2013

Part I. Introduction

- What is optimization?
- Applications in Operations Research, Economics, Statistics, Game Theory, ...
- Inverse optimization

What is optimization?

- **In mathematics:** optimization is a theory on maximization or minimization of functions defined on special sets.
  - We are given an **objective function** $f : \mathbb{R}^n \to \mathbb{R}$ which is to be maximized/minimized over a given region $\Omega \subseteq \mathbb{R}^n$, called **feasible region**.
- **In applications:** we are always maximizing or minimizing something, for example:
  - **Microeconomics:** we want to maximize profit of a firm or minimize its costs under limited resources;
  - **Statistics:** we want to maximize likelihood or minimize residual error (e.g. residual sum of squares);
  - **Experimental Design:** we want to maximize measurement precision in an experiment under restricted possibilities of available laboratory equipment;
  - **Portfolio Theory:** we want to maximize return of an investment under budget constraints and regulatory constraints;
  - **Operations Research:** we want to minimize the length of the path of the Traveling Salesman;
  - many more applications in engineering, physics, chemistry etc.

A basic example: Nutrition (Diet) Problem

In practice, this problem is solved e.g. by producers of dog food, often on a daily basis.

- A producer of dog food processes leftovers from butchers, slaughterhouses and meat-processing plants.
- The producer must combine the available raw materials to achieve the declared nutrition content, e.g. enough proteins, enough calories, not too much salt, not too much fat etc.
- The producer does not care about what the ingredients exactly are:
  - (s)he simply buys anything from which it is possible to combine the declared nutrition levels, as cheaply as possible,
  - (s)he mixes and boils the raw materials, getting a homogenous tasteless mesh,
  - (s)he adds meat perfume,
  - (s)he fills the 'product' into cans,
  - (s)he adds one piece of real meat just under the cover of each can (for a better visual effect),
  - (s)he spends plenty of money on marketing to be able to sell this stuff.

A basic example: Nutrition Problem (continued)

For example assume that a meat processing plant offers two kinds of leftovers:

- $x_1 =$ leftover of salami,
- $x_2 =$ mechanically separated meat.

Now we can summarize data for our problem: we know the contents of proteins, fat, and salt in each of the two ingredients x_1, x_2, and we know their prices per ton.

Our goal. We want to mix them to obtain dog food containing at least a declared level of proteins, at most a declared level of fat and at most a declared level of salt. Furthermore, we want to minimize costs.

A basic example: Nutrition Problem (continued)

We get the following optimization problem:

	salami		sep.meat		demand
minimize	$c_1 x_1$	$+$	$c_2 x_2$		
subject to:					
proteins	$a_{11} x_1$	$+$	$a_{12} x_2$	$\geqslant$	b_1
fat	$a_{21} x_1$	$+$	$a_{22} x_2$	$\leqslant$	b_2
salt	$a_{31} x_1$	$+$	$a_{32} x_2$	$\leqslant$	b_3
	x_1			$\geqslant$	0
			x_2	$\geqslant$	0

Data of the optimization problem are denoted in red:

- a_{ij} denote the contents of proteins, fat, salt in one tone of salami and separated meat,
- b_i denote the demands,
- c_j denote the prices per ton of salami and separated meat.

Examples of optimization problems — nonlinear case

- Portfolio optimization. Data: C, r, γ. Find a portfolio with average yield $\geqslant \gamma$ and minimal variance:

$$\min_x x^{\mathrm{T}} C x \quad \text{s.t.} \quad \sum_i x_i = 1, x \geqslant 0, \sum_i r_i x_i \geqslant \gamma.$$

- Logistic regression. Data: observed pairs $[x_1, y_1], \ldots, [x_n, y_n]$. Find least-squares estimates of regression parameters $\beta_1, \beta_2, \beta_3$:

$$\min_{\beta_1,\beta_2,\beta_3} \sum_{i=1}^{n} \left[y_i - \frac{\beta_1}{1 + e^{-\beta_2(x_i - \beta_3)}} \right]^2 \quad \text{s.t.} \quad \beta_1 \geqslant 0, \beta_2 \geqslant 0.$$

- Smallest-volume circumscribing ellipsoid. Data: points $x_1, \ldots, x_n \in \mathbb{R}^n$. Solve

$$\min_{E \in \mathbb{R}^{n\times n}, s \in \mathbb{R}^n} \det E \quad \text{s.t.} \quad (\forall i)\ (x_i - s)^{\mathrm{T}} E^{-1} (x_i - s) \leqslant 1,\ \ E \text{ p.s.d.}$$

Part II. Inverse optimization

- General formulation
- Questions and problems
- Optimal value function
- Applications

General formulation

Let $\Theta \subseteq \mathbb{R}^k$ (admissible region) be given. Consider the class of optimization problems

$$\min_x \{\varphi(x;\theta) : \; g_1(x;\theta) \leqslant 0, \; \ldots, \; g_m(x;\theta) \leqslant 0, \; x \in \mathbb{R}^n\} \quad \theta \in \Theta, \qquad (1)$$

where

- $x \in \mathbb{R}^n$: vector of variables,
- $\theta \in \mathbb{R}^k$: data vector,
- $\varphi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$: objective function,
- $g_1, \ldots, g_m : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$: constraint functions.

When we fix $\theta_0 \in \Theta$, we say that we select the scenario θ_0, or, that we select the optimization problem

$$\min_x \varphi(x;\theta_0) \;\; \text{s.t.} \;\; g_1(x;\theta_0) \leqslant 0, \; \ldots, \; g_m(x;\theta_0) \leqslant 0$$

from the family (1).

Inverse optimization problem: given $\lambda_0 \in \mathbb{R}$, find $\theta_0 \in \Theta$ such that

$$\min_x \{\varphi(x;\theta_0) : \; g_1(x;\theta_0) \leqslant 0, \; \dots, \; g_m(x;\theta_0) \leqslant 0\} = \lambda_0$$

or assert that none exists.

Interpretation. The data of the inverse optimization problem consist of the functions $\varphi, g_1, \dots, g_m$, the set Θ and the value λ_0, called demand. We can say that we are to 'design' an optimization problem

$$\min_x \{\varphi(x;\theta_0) : g_1(x;\theta_0) \leqslant 0, \dots, g_m(x;\theta_0) \leqslant 0\}$$

(or: 'select a scenario') attaining the prescribed optimal value λ_0. Our constraints are that we are allowed to select the parameter vector θ_0 only from the admissible set Θ.

Special case: Inverse linear programming

- **Traditional linear programming.**
  - **Data:** $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.
  - **Task:** Find $x \in \mathbb{R}^n$ solving $\min c^{\mathrm{T}}x$ s.t. $Ax = b, x \geqslant 0$.
- **Inverse linear programming.**
  - **Data:** $\Theta \subseteq \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}$.
  - **Task:** Find $(A, b, c) \in \Theta$ such that $\min\{c^{\mathrm{T}}x : \ Ax = b, x \geqslant 0\} = \lambda_0$.
- **Inverse linear programming with interval coefficients.**
  - **Data:** $\mathbf{A} \in \mathbb{IR}^{m\times n}$, $\mathbf{b} \in \mathbb{IR}^m$, $\mathbf{c} \in \mathbb{IR}^n$ and $\lambda_0 \in \mathbb{R}$.
  - **Task:** Find $(A, b, c) \in \Theta := (\mathbf{A}, \mathbf{b}, \mathbf{c})$ such that $\min\{c^{\mathrm{T}}x : \ Ax = b, x \geqslant 0\} = \lambda_0$.
  - Here: $\mathbb{IR}^{m\times n}$ is the space of all interval matrices. An interval matrix is a family of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leqslant A \leqslant \overline{A}\},$$

where "$\leqslant$" is understood componentwise.

Questions and problems

From now on: we will study only the case of inverse LP with interval coefficients ("IILP"). Of course, the questions and problems apply more generally.

Solution space. Let Θ^* denote the set of solutions to IILP, i.e.

$$\Theta^* = \big\{(A,b,c) \in (\mathbf{A},\mathbf{b},\mathbf{c}) : \min_x \{c^{\mathrm{T}} x : Ax = b, x \geqslant 0\} = \lambda_0\big\}.$$

Some natural questions.

- Can we test whether $\Theta^* \neq \emptyset$? (I.e., can we test whether the problem has at least one solution?) Can we decide by an efficient algorithm (i.e., in polynomial time), or is the problem computationally hard (say, NP-hard)?
- Can we test whether Θ^* is a singleton? (I.e., can we test uniqueness of the solution?)
- How to find some $\theta^* \in \Theta^*$?
- How to describe or approximate the set Θ^* if it is intricate?
- How to determine further set-theoretic properties of Θ^*, such as connectivity, (un)boundedness etc.?

Questions and problems (contd.)

Optimal value function:

$$f(A,b,c) = \inf_x \{c^{\mathrm{T}}x : Ax = b, x \geqslant 0\}.$$

Remark.

- $f(A,b,c) = -\infty$ means that the scenario (A,b,c) is unbounded.
- $f(A,b,c) = \infty$ means that the scenario (A,b,c) is infeasible.

Some natural questions:

- Is the optimal value function f continuous on $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$?
- Is the optimal value function f monotone on $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$?
- Does it hold $f(A,b,c) = \infty$ for some A,b,c?
- Does it hold $f(A,b,c) = -\infty$ for some A,b,c?
- More generally: how to describe the range of $f(A,b,c)$ over the domain $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$?

Two applications

Example 1. Finding a maximal flow in a network can be written as a linear programming problem.

- Data: capacities of edges.
- IILP: given possible intervals for capacities of edges, find the capacities in order to achieve the prescribed maximal flow.

Example 2. Designing a matrix game.

- Data: payoff matrix.
- IILP: given an interval of admissible payoff matrices, find the payoff matrix with a prescribed value of the game.
- Remark. Recall that finding the Nash mixed strategy can be solved via the linear program

$$\max_{\gamma,x} \gamma \quad \text{s.t.} \quad Ax \geqslant \gamma e, x \geqslant 0, e^{\mathrm{T}} x = 1,$$

where $e = (1, \ldots, 1)^{\mathrm{T}}$ and A is the payoff matrix.

Part III. Some theory of IILP

- Continuity of the optimal value function
- Binary Search
- Parametric programming

Some theory of IILP

Two observations:

- The optimal value function $f(A, b, c) = \min\{c^{\mathrm{T}}x : Ax = b, x \geqslant 0\}$ is computable in polynomial time ("easy-to-evaluate"), using e.g. Interior Point Methods.
- The admissible space $\Theta = \mathbf{A} \times \mathbf{b} \times \mathbf{c}$ is a convex set.

Crucial questions:

- Is the optimal value function f continuous?
- Given λ_0,
  - (lower bound): can we find a scenario $(A_0, b_0, c_0) \in (\mathbf{A}, \mathbf{b}, \mathbf{c})$ such that $f(A_0, b_0, c_0) < \lambda_0$?
  - (upper bound): can we find a scenario $(A_1, b_1, c_1) \in (\mathbf{A}, \mathbf{b}, \mathbf{c})$ such that $f(A_1, b_1, c_1) > \lambda_0$?

If all answers are positive, we can use Binary Search: by convexity of $\mathbf{A} \times \mathbf{b} \times \mathbf{c}$ we can define

$$v(\mu) = f((1-\mu)A_0 + \mu A_1, (1-\mu)b_0 + \mu b_1, (1-\mu)c_0 + \mu c_1) - \lambda_0$$

and using fast computability we simply find its root over $\mu \in [0, 1]$.

Binary search

To recall:

$$v(\mu) = f((1-\mu)A_0 + \mu A_1, (1-\mu)b_0 + \mu b_1, (1-\mu)c_0 + \mu c_1) - \lambda_0.$$

Binary Search:

(1) input: precision parameter $\varepsilon > 0$, λ_0, (A_0, b_0, c_0), (A_1, b_1, c_1).

(2) set $\underline{\mu} := 0$, $\overline{\mu} := 1$

(3) set $\mu' := \frac{1}{2}(\underline{\mu} + \overline{\mu})$

(4) if $|v(\mu') - \lambda_0| < \varepsilon$ then return the scenario $((1-\mu')A_0 + \mu' A_1, (1-\mu')b_0 + \mu' b_1, (1-\mu')c_0 + \mu' c_1)$ and terminate

(5) if $v(\mu') < \lambda_0$ then set $\underline{\mu} := \mu'$

(6) if $v(\mu') > \lambda_0$ then set $\overline{\mu} := \mu'$

(7) go to 3.

Is the optimal value function continuous?

Problem: the optimal value function
$f(A,b,c) = \min\{c^{\mathrm{T}}x : Ax = b, x \geqslant 0\}$ need not be continuous.

Theorem. If every scenario $(A,b,c) \in (\mathbf{A},\mathbf{b},\mathbf{c})$ satisfies

(a) $\{x : Ax = 0, x \geqslant 0, c^{\mathrm{T}}x \leqslant 0\} = \{0\}$,

(b) $\{y : A^{\mathrm{T}}y \leqslant 0, b^{\mathrm{T}}y \geqslant 0\} = \{0\}$,

then f is continuous.

Theorem. The condition (a) is satisfied iff the linear programming problem

$$\underline{A}x \leqslant 0,\ \overline{A}x \geqslant 0,\ x \geqslant 0,\ \underline{c}^{\mathrm{T}}x \leqslant 0,\ \sum_i x_i = 1$$

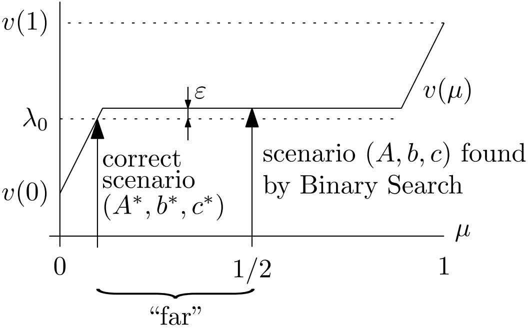
is infeasible.

Corollary. Testing whether (a) holds can be done in polynomial computation time.

Unfortunately: Testing (b) is NP-hard. But this would be another story...

A drawback of Binary Search

Binary Search finds the desired scenario only approximately (depending on the precision parameter $\varepsilon > 0$), meaning that it can make a significant error:



Parametric programming technique

To recall: an index set $B = \{i_1, \dots, i_n\}$ is a basis if A_B is nonsingular. (If a_i is i-th column of A, then A_B has columns $a_{i_1}, \dots, a_{i_n}$.)

Roughly: a basis B is optimal if $x = A_B^{-1} b$ is an optimal solution of the linear programming problem $\min\{c^{\mathrm{T}} x : Ax = b, x \geqslant 0\}$.

We denote $R := \{1, \dots, m\} \setminus B$ and write A_R accordingly.

A well-known theorem from LP. A basis B is optimal iff the following conditions hold:

- feasibility condition: $A_B^{-1} b \geqslant 0$,
- optimality condition: $c_R - c_B A_B^{-1} A_R \geqslant 0$.

Parametric programming technique (contd.)

The main ingredient — "a shift from scenario $A(\kappa_0), b(\kappa_0), c(\kappa_0)$ to scenario $A(\kappa_1), b(\kappa_1), c(\kappa_1)$".

- Assume that A, b, c depend on a parameter κ and write $A(\kappa), b(\kappa), c(\kappa)$.
- Let $\kappa_0 < \kappa_1$ be given. Let B be an optimal basis for $\min\{c(\kappa_0)^{\mathrm{T}}x : A(\kappa_0)x = b(\kappa_0), x \geqslant 0\}$.
- Find

$$\kappa^* = \min\{\kappa_1, \sup\{\kappa : B \text{ is an optimal basis for } \min\{c(\kappa)^{\mathrm{T}}x : A(\kappa)x = b(\kappa), x \geqslant 0\}\}\}. \quad (2)$$

  (Remark: we find the maximal κ such that both the feasibility condition $A(\kappa)_B^{-1}b(\kappa) \geqslant 0$ and the optimality condition $c(\kappa)_R - c(\kappa)_B A(\kappa)_B^{-1}A(\kappa)_R \geqslant 0$ hold.)
- If $\kappa^* < \kappa_1$, then $\min\{c(\kappa^*)^{\mathrm{T}}x : A(\kappa^*)x = b(\kappa^*), x \geqslant 0\}$ must have another optimal basis B. So, find it and repeat (2). Stop when $\kappa^* = \kappa_1$.

Parametric programming technique (contd.)

Crucial issue: how to find $\kappa^* = \sup\{\kappa : B$ is an optimal basis for $\min\{c(\kappa)^{\mathrm{T}}x : A(\kappa)x = b(\kappa), x \geqslant 0\}\}$?

Without details: we have the following theorem.

- If $A(\kappa) = A$, $b(\kappa) = b$ and $c(\kappa) = (1-\kappa)c_0 + \kappa c_1$, then κ^* can be found in polynomial time.
- If $A(\kappa) = A$, $b(\kappa) = (1-\kappa)b_0 + \kappa b_1$ and $c(\kappa) = c$, then κ^* can be found in polynomial time.
- Rank-one lemma: If $A(\kappa) = A_0 + \kappa A^*$, where A^* has rank one, $b(\kappa) = b$ and $c(\kappa) = c$, then κ^* can be found in polynomial time.

Parametric programming technique (contd.)

Now we can roughly describe the parametric programming method.

- Let the optimal value function be continuous.
- Let (A_0, b_0, c_0) be a scenario s.t. $\min\{c_0^{\mathrm{T}} x : A_0 x = b_0, x \geqslant 0\} < \lambda_0$.
- Let (A_1, b_1, c_1) be a scenario s.t. $\min\{c_1^{\mathrm{T}} x : A_1 x = b_1, x \geqslant 0\} > \lambda_0$.
- Stage I. Set $A = A_0$, $b = b_0$, $c = (1-\kappa)c_0 + \kappa c_1$ and shift $(A_0, b_0, c_0) \to (A_0, b_0, c_1)$.
- Stage II. Set $A = A_0$, $b = (1-\kappa)b_0 + \kappa b_1$, $c = c_1$ and shift $(A_0, b_0, c_1) \to (A_0, b_1, c_1)$.
- Stage III.
  - Choose a rank-one decomposition: choose a sequence of rank-one matrices $A_1^*, \ldots, A_\ell^*$ such that $A_0 + A_1^* + \cdots, A_\ell^* = A_1$.
  - Make shifts

$$(A_0, b_1, c_1) \to (A_0 + A_1^*, b_1, c_1) \to (A_0 + A_1^* + A_2^*, b_1, c_1) \to$$
$$\to \cdots \to (A_0 + A_1^* + A_2^* + \cdots + A_\ell^*, b_1, c_1) = (A_1, b_1, c_1).$$

Conclusions: pros and cons

Binary Search:

- $(+)$ in practice: often fast
- $(-)$ only ε-exact solution is found
- $(-)$ it is hard to find a theoretical bound on the number of iterations to achieve ε-convergence

Parametric Programming Approach:

- $(-)$ in practice: usually slower than Binary Search
- $(+)$ exact solution is found
- $(+)$ more flexible (a user can choose what will be perturbed first)
- $(-)$ worst-case complexity can be bad (similar to the Simplex Algorithm)

Both approaches:

- $(-)$ continuity of the optimal value function is required
- $(-)$ a-priori knowledge of (A_0, b_0, c_0) and (A_1, b_1, c_1) is required

Thank You! And... some further reading...

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