Characterizing and bounding eigenvalues of interval matrices

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Why intervals?

Use intervals

- to handle uncertainty
- for their simplicity – just lower and upper bound
- when dealing with continuum of states
- for sensitivity analysis
- for reliable results

State-of-the-art

Enclosures for eigenvalues of interval matrices

- Franzè, Carotenuto and Balestrino (2006):
  Gerschgorin-like regions for locating eigenvalues
- Mayer (1994):
  an enclosure method for eigenvalues based on Taylor expansion
- Ahn, Moore and Chen (2006):
  an estimation on eigenvalues based on perturbation theory
- Rohn (1998):
  a cheap formula for an eigenvalue enclosure
- Kolev (2006):
  outer estimation for general case with non-linear dependencies
- H., Daney and Tsigaridas (2010):
  several formulae for eigenvalue enclosures

- Plenty of papers on Schur/Hurwitz stability of interval matrices

State-of-the-art

Theoretical results

Early works:

- Soh (1990)
- Deif (1991)
- Hertz (1992)
- Deif & Rohn (1992, 1994)
- Rohn (1993)

- Few novel works.

Introduction

Notation – interval quantities

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n} \mid \underline{A} \le A \le \overline{A}\}.$$

The midpoint and the radius of $\mathbf{A}$

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Objective

Determine / enclose the eigenvalue set

$$\Lambda(\mathbf{A}) := \{\lambda \in \mathbb{C} \mid Ax = \lambda x \text{ for some } A \in \mathbf{A}, x \neq 0\}.$$

Introduction

Definition

A symmetric interval matrix

$$\mathbf{A}^S := \{A \in \mathbf{A} \mid A = A^T\}.$$

Definition

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Denote its (real) eigenvalues

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Objective

Determine / enclose the eigenvalue sets

$$\boldsymbol{\lambda}_i(\mathbf{A}^S) := \{\lambda_i(A) \mid A \in \mathbf{A}^S\}, \quad i = 1, \ldots, n.$$

Symmetric case

Theorem (Rohn, 2005)

We have

$$\boldsymbol{\lambda}_i(\mathbf{A}^S) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)].$$

Properties

- Easy and cheap to compute;
- good starting point;
- all intervals the same width.

Other approaches

- Interlacing method;
- filtering method;
- ...

Symmetric case

Theorem (Hertz, 1992)

We have

$$\overline{\lambda}_1(\mathbf{A}^S) = \max_{z\in\{\pm1\}^n} \lambda_1(A_c + \mathrm{diag}(z)\, A_\Delta\, \mathrm{diag}(z)),$$

$$\underline{\lambda}_n(\mathbf{A}^S) = \min_{z\in\{\pm1\}^n} \lambda_n(A_c - \mathrm{diag}(z)\, A_\Delta\, \mathrm{diag}(z)).$$

Theorem (H., Daney, Tsigaridas, 2011)

If $\lambda \in \partial\Lambda(\mathbf{A}^S)$, then $\mathbf{A}^S$ has a principal submatrix $\hat{\mathbf{A}}^S$ of size k such that:

- *If $\lambda = \overline{\lambda}_j(\mathbf{A}^S)$ for some $j \in \{1,\dots,n\}$, then*

$$\lambda \in \big\{\lambda_i\big(\hat{A}_c + \mathrm{diag}(z)\,\hat{A}_\Delta\,\mathrm{diag}(z)\big);\ z \in \{\pm1\}^k,\ i = 1,\dots,k\big\}.$$

- *If $\lambda = \underline{\lambda}_j(\mathbf{A}^S)$ for some $j \in \{1,\dots,n\}$, then*

$$\lambda \in \big\{\lambda_i\big(\hat{A}_c - \mathrm{diag}(z)\,\hat{A}_\Delta\,\mathrm{diag}(z)\big);\ z \in \{\pm1\}^k,\ i = 1,\dots,k\big\}.$$

General case

Theorem

Let A_c be diagonalizable, i.e., $V^{-1}A_cV$ is diagonal for some $V \in \mathbb{C}^{n\times n}$. Then for every $\lambda \in \Lambda(\mathbf{A})$ there is $i \in \{1,\dots,n\}$ such that

$$|\lambda - \lambda_i(A_c)| \leq \kappa_2(V) \cdot \sigma_{\max}(A_\Delta).$$

Proof.

Use the Bauer & Fike Theorem (1960). □

General case

Theorem

Let $V^{-1}A_cV = J$ *be the Jordan canonical form of* A_c*, and let* p *be the maximal dimension of the Jordan's blocks in* J*. Denote*

$$\Theta_2 = \sqrt{\frac{p(p+1)}{2}} \cdot \kappa_2(V) \cdot \sigma_{\max}(A_\Delta), \quad \Theta = \max\{\Theta_2, \Theta_2^{\frac{1}{p}}\}.$$

Then for every $\lambda \in \Lambda(\mathbf{A})$ *there is* $i \in \{1, \dots, n\}$ *such that*

$$|\lambda - \lambda_i(A_c)| \le \Theta.$$

Proof.

Use Chu's Theorem (1986). □

General case

Theorem (Rohn, 1998)

For each eigenvalue $\lambda_r + i\lambda_i \in \Lambda(\mathbf{A})$ *we have*

$$\lambda_r \le \lambda_1\left(\frac{1}{2}(A_c + A_c^T)\right) + \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),$$

$$\lambda_r \ge \lambda_n\left(\frac{1}{2}(A_c + A_c^T)\right) - \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),$$

$$\lambda_i \le \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} + \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix},$$

$$\lambda_i \ge \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} - \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix}.$$

General case

Theorem

We have that $A(x+iy) = (\lambda_r + i\lambda_i)(x+iy)$ *for some* $A \in \mathbf{A}$ *iff*

$$\begin{aligned} A_\Delta|x| &\geq |A_c x - \lambda_r x + \lambda_i y|, \\ A_\Delta|y| &\geq |A_c y - \lambda_r y - \lambda_i x|, \\ A_\Delta|xy^T - yx^T| &\geq |(A_c x - \lambda_r x + \lambda_i y)y^T - (A_c y - \lambda_r y - \lambda_i x)^T x|. \end{aligned}$$

Properties

- Description by means of $n^2 + 2n$ nonlinear inequalities;
- Its shape represented piecewise by quadrics.

Complex case

Theorem (Hladík, 2011)

Let $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{n\times n}$. *Then for each eigenvalue* $\lambda_r + i\lambda_i \in \Lambda(\mathbf{A} + i\mathbf{B})$ *we have*

$$\underline{\lambda}_n \begin{pmatrix} \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) & \frac{1}{2}(\mathbf{B}^T-\mathbf{B}) \\ \frac{1}{2}(\mathbf{B}-\mathbf{B}^T) & \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) \end{pmatrix}^S \le \lambda_r \le \overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) & \frac{1}{2}(\mathbf{B}^T-\mathbf{B}) \\ \frac{1}{2}(\mathbf{B}-\mathbf{B}^T) & \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) \end{pmatrix}^S,$$

$$\underline{\lambda}_n \begin{pmatrix} \frac{1}{2}(\mathbf{B}+\mathbf{B}^T) & \frac{1}{2}(\mathbf{A}-\mathbf{A}^T) \\ \frac{1}{2}(\mathbf{A}^T-\mathbf{A}) & \frac{1}{2}(\mathbf{B}+\mathbf{B}^T) \end{pmatrix}^S \le \lambda_i \le \overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\mathbf{B}+\mathbf{B}^T) & \frac{1}{2}(\mathbf{A}-\mathbf{A}^T) \\ \frac{1}{2}(\mathbf{A}^T-\mathbf{A}) & \frac{1}{2}(\mathbf{B}+\mathbf{B}^T) \end{pmatrix}^S.$$

Properties

- It reduces the problem to the symmetric case.
- It generalizes Rohn's Theorem (1998)

Examples

Example (Petkovski, 1991)

Let

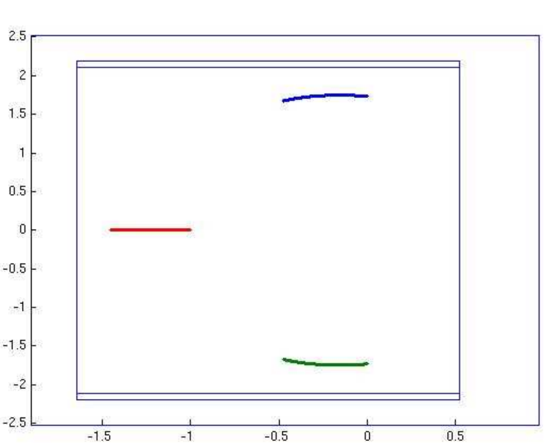
$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 2 & [-1.399, -0.001] & 0 \\ 1 & 0.5 & -1 \end{pmatrix}$$

Bounds by our approach using

- Rohn's theorem (2005):
  $$\lambda \in [-1.9068, 0.9702], \;\; \mu \in [-2.5191, 2.5191].$$
- filtering (2011):
  $$\lambda \in [-1.6474, 0.5205], \;\; \mu \in [-2.1934, 2.1934].$$
- best bounds by Hertz (1992):
  $$\lambda \in [-1.6474, 0.5205], \;\; \mu \in [-2.1112, 2.1112].$$

Example (con't)

Monter Carlo simulation:



Example (Xiao and Unbehauen, 2000)

Let

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & [-1,1] \\ 0 & -1 & [-1,1] \\ [-1,1] & [-1,1] & 0.1 \end{pmatrix}$$

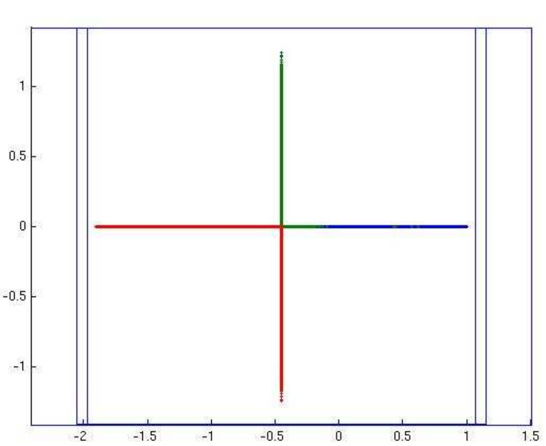
Bounds by our approach using

- Rohn's theorem (2005):
  $$\lambda \in [-2.4143, 1.5143], \;\; \mu \in [-1.4143, 1.4143].$$
- filtering:
  $$\lambda \in [-2.0532, 1.1532], \;\; \mu \in [-1.4143, 1.4143].$$
- best bounds by Hertz (1992):
  $$\lambda \in [-1.9674, 1.0674], \;\; \mu \in [-1.4143, 1.4143].$$

Examples

Example (con't)

Monter Carlo simulation:



Examples

Example (Wang, Michel and Liu, 1994)

Let

$$\mathbf{A} = \begin{pmatrix} [-3,-2] & [4,5] & [4,6] & [-1,1.5] \\ [-4,-3] & [-4,-3] & [-4,-3] & [1,2] \\ [-5,-4] & [2,3] & [-5,-4] & [-1,0] \\ [-1,0.1] & [0,1] & [1,2] & [-4,2.5] \end{pmatrix}$$

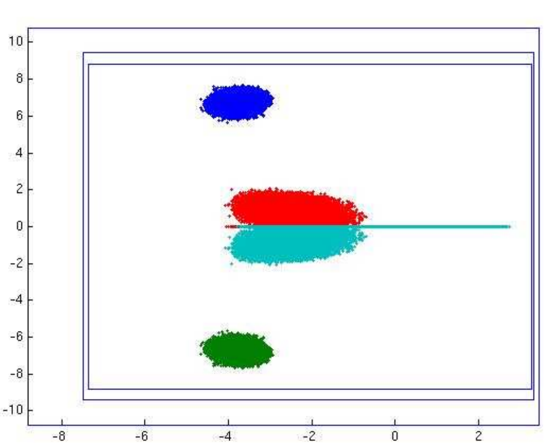
Bounds by our approach using

- Rohn's theorem (2005):
  $$\lambda \in [-8.8221, 3.4408], \ \ \mu \in [-10.7497, 10.7497].$$
- filtering:
  $$\lambda \in [-7.4848, 3.3184], \ \ \mu \in [-9.4224, 9.4224].$$
- best bounds by Hertz (1992):
  $$\lambda \in [-7.3691, 3.2742], \ \ \mu \in [-8.7948, 8.7948].$$

Examples

Example (con't)

Monter Carlo simulation:



Examples

Example (Seyranian, Kirillov, and Mailybaev, 2005)

For $s \in \mathbf{s} = ([0, 0.2], [0.9797, 1], 0)$ let

$$B(s) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} + i \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 2 \\ 4 & 2 & 0 \end{pmatrix} + 4i \begin{pmatrix} 0 & -s_1 - i\, s_2 & i\, s_3 \\ s_1 + i\, s_2 & 0 & -s_3 \\ -i\, s_3 & s_3 & 0 \end{pmatrix}.$$

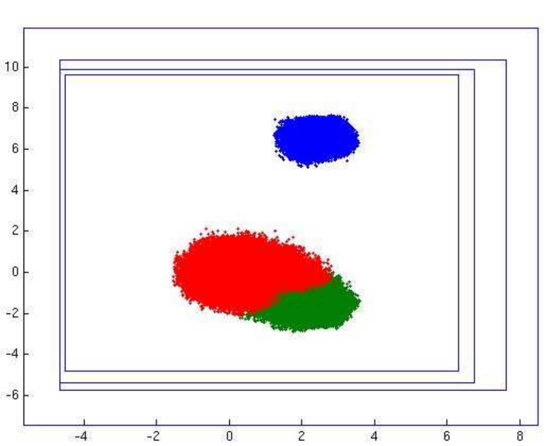
Bounds by our approach using

- Rohn's theorem (2005):
  $\lambda \in [-4.6546, 7.6146], \;\; \mu \in [-5.7632, 10.3387].$
- filtering:
  $\lambda \in [-4.6546, 6.7421], \;\; \mu \in [-5.4017, 9.8787].$
- best bounds by Hertz (1992):
  $\lambda \in [-4.5180, 6.3031], \;\; \mu \in [-4.8237, 9.6227].$

(The overall box by Hertz (2009))

Example (con't)

Monter Carlo simulation:



The End

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