New Directions in Interval Linear Programming

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Objectives

Objectives of the presentation

To show that interval linear programming

- has important applications
- has many nice results
- has challenging problems

Outline

1. Interval linear programming introduction
    - interval linear inequalities
    - complexity issues
2. Interval linear programming problems
    - optimal value range
    - optimal solution set
3. Interval linear programming applications
    - interval linear regression
    - constraint programming and global optimization

Interval linear inequalities

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The midpoint and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Theorem (Oettli–Prager, 1964)

A vector $x \in \mathbb{R}^n$ is a solution of $\mathbf{A}x = \mathbf{b}$ if and only if

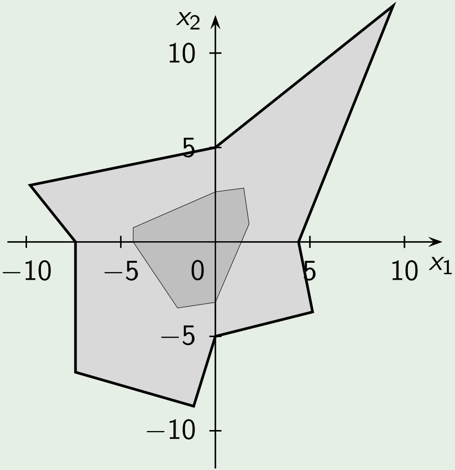
$$|A_c x - b_c| \leq A_\Delta |x| + b_\Delta.$$

Theorem (Gerlach, 1981)

A vector $x \in \mathbb{R}^n$ is a solution of $\mathbf{A}x \leq \mathbf{b}$ if and only if

$$A_c x - b_c \leq A_\Delta |x| + b_\Delta.$$

Example (An interval polyhedron)



$$\begin{pmatrix} -[2,5] & -[7,11] \\ [1,13] & -[4,6] \\ [5,8] & [-2,1] \\ -[1,4] & [5,9] \\ -[5,6] & -[0,4] \end{pmatrix} x \leq \begin{pmatrix} [61,63] \\ [19,20] \\ [15,22] \\ [24,25] \\ [26,37] \end{pmatrix}$$

- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,

Interval linear programming

Linear programming

Three basic forms of linear programs

$$f(A,b,c) \equiv \min c^T x \ \text{ subject to } \ Ax = b,\ x \geq 0,$$
$$f(A,b,c) \equiv \min c^T x \ \text{ subject to } \ Ax \leq b,$$
$$f(A,b,c) \equiv \min c^T x \ \text{ subject to } \ Ax \leq b,\ x \geq 0.$$

Interval linear programming

Family of linear programs with $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, in short

$$f(\mathbf{A},\mathbf{b},\mathbf{c}) \equiv \min \mathbf{c}^T x \ \text{ subject to } \ \mathbf{A}x \overset{(\leq)}{=} \mathbf{b},\ (x \geq 0).$$

The three forms are not transformable between each other!

Goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

Complexity of basic problems

	$\mathbf{A}x = \mathbf{b},\ x \geq 0$	$\mathbf{A}x \leq \mathbf{b}$	$\mathbf{A}x \leq \mathbf{b},\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	suff. / necessary conditions only	suff. / necessary conditions only	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	suff. / necessary conditions only	suff. / necessary conditions only	suff. / necessary conditions only
optimal value range	$\underline{f}$ polynomial $\overline{f}$ NP-hard	$\underline{f}$ NP-hard $\overline{f}$ polynomial	polynomial

Optimal value range

Optimal value range

Definition

$$\underline{f} := \min f(A,b,c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c},$$
$$\overline{f} := \max f(A,b,c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}.$$

Theorem (Rohn, 2006)

We have for type ($\mathbf{A}x = \mathbf{b},\ x \geq 0$)

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0,$$
$$\overline{f} = \max_{p \in \{\pm 1\}^m} f(A_c - \mathrm{diag}(p)\, A_\Delta, b_c + \mathrm{diag}(p)\, b_\Delta, \overline{c}).$$

Theorem (Vajda, 1961)

We have for type ($\mathbf{A}x \leq \mathbf{b},\ x \geq 0$)

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ x \geq 0,$$
$$\overline{f} = \min \overline{c}^T x \ \textit{ subject to } \ \overline{A}x \leq \underline{b},\ x \geq 0.$$

Optimal value range

Algorithm (Optimal value range $[\underline{f}, \overline{f}]$)

1. Compute

$$\underline{f} := \inf\ c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ x \in \mathcal{M},$$

   where $\mathcal{M}$ is the primal solution set.

2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.

3. Compute

$$\overline{\varphi} := \sup\ b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ y \in \mathcal{N},$$

   where $\mathcal{N}$ is the dual solution set.

4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.

5. If the primal problem is strongly feasible, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

Optimal solution set

Optimal solution set

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \mathbf{A},\, b \in \mathbf{b},\, c \in \mathbf{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Optimal solution set

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, and $c \in \mathbf{c}$ such that

$$Ax = b,\ x \geq 0,\ A^T y \leq c,\ c^T x = b^T y,$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Relaxation

Relaxing the dependencies

$$\mathbf{A}x = \mathbf{b},\ x \geq 0,\ \mathbf{A}^T y \leq \mathbf{c},\ \mathbf{c}^T x = \mathbf{b}^T y,$$

which is described by

$$\underline{A}x \leq \overline{b},\quad -\overline{A}x \leq -\underline{b},\quad x \geq 0,$$
$$A_c^T y - A_\Delta^T |y| \leq \overline{c},\quad |c_c^T x - b_c^T y| \leq c_\Delta^T x + b_\Delta^T |y|.$$

Linearization of $|y|$

Properties

- The solution set is non-convex in general
- It is linear at any orthant
- NP-hard to obtain exact bounds

Theorem (Beaumont, 1998)

For every $y \in \mathbf{y} \subset \mathbb{R}$ with $\underline{y} < \overline{y}$ one has

$$|y| \leq \alpha y + \beta, \tag{1}$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \textit{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \geq 0$ or $\overline{y} \leq 0$ then (1) holds as equation.

Linearization of $|y|$

Now, the linearization reads

$$
\begin{aligned}
\underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,\\
\big(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)\big)y \le \overline{c} + A_\Delta^T \beta,\\
\underline{c}^T x + \big(-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T \beta,\\
-\overline{c}^T x + \big(b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T \beta,
\end{aligned}
$$

where

$$
\alpha_i := \begin{cases} \frac{|\overline{y}_i| - |\underline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i,\\ \operatorname{sgn}(\overline{y}_i) & \text{if } \underline{y}_i = \overline{y}_i, \end{cases}
$$

$$
\beta_i := \begin{cases} \frac{\overline{y}_i |\underline{y}_i| - \underline{y}_i |\overline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i,\\ 0 & \text{if } \underline{y}_i = \overline{y}_i. \end{cases}
$$

Contractor

Algorithm (Optimal solution set contractor)

1. Compute an initial interval enclosure $\mathbf{x}^0, \mathbf{y}^0$
2. $i := 0$;
3. **repeat**
    1. compute the interval hull $\mathbf{x}^i, \mathbf{y}^i$ of the linearized system;
    2. $i := i + 1$;
4. **until** improvement is nonsignificant;
5. **return** $\mathbf{x}^i$;

Properties

- Each iteration requires computing the interval hull ($2(m+n)$ linear programs).
- In practice, it converges quickly, but not to $\mathcal{S}$ in general.

Example

Example

Consider an interval linear program

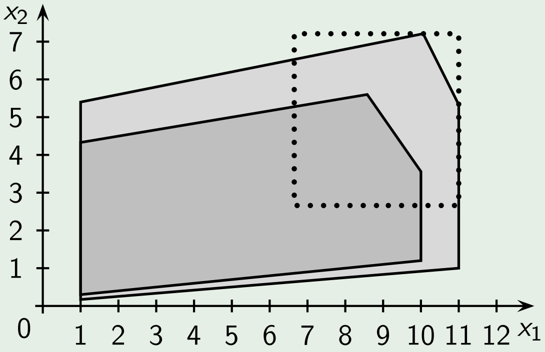
$$
\begin{aligned}
\min -[15,16]x_1 - [17,18]x_2 \ \ &\text{subject to} \\
x_1 &\leq [10,11], \\
-x_1 + [5,6]x_2 &\leq [25,26], \\
[6,6.5]x_1 + [3,4.5]x_2 &\leq [81,82], \\
-x_1 &\leq -1, \\
x_1 - [10,12]x_2 &\leq -[1,2].
\end{aligned}
$$

Take the initial enclosure

$$
\begin{aligned}
\mathbf{x}^0 &= 1000 \cdot ([-1,1],\ [-1,1])^T, \\
\mathbf{y}^0 &= 1000 \cdot ([0,1],\ [0,1],\ [0,1],\ [0,1],\ [0,1])^T.
\end{aligned}
$$

Example

Example (cont.)



- Only four iterations needed.
- In grey the largest and the smallest feasible area.
- The final enclosure of the optimal solution set $\mathcal{S}$ is dotted.

Basis stability

Definition

The interval linear programming problem

$$\min \mathbf{c}^T x \;\text{ subject to }\; \mathbf{A}x = \mathbf{b},\; x \geq 0,$$

is B-stable if B is an optimal basis for each realization.

Theorem

B-stability implies that the optimal value bounds are

$$\begin{aligned}
\underline{f} &= \min\; \underline{c}_B^T x \;\text{ subject to }\; \underline{A}_B x_B \leq \overline{b},\; -\overline{A}_B x_B \leq -\underline{b},\; x_B \geq 0,\\
\overline{f} &= \max\; \overline{c}_B^T x \;\text{ subject to }\; \underline{A}_B x_B \leq \overline{b},\; -\overline{A}_B x_B \leq -\underline{b},\; x_B \geq 0.
\end{aligned}$$

Under the unique B-stability, the set of all optimal solutions reads

$$\underline{A}_B x_B \leq \overline{b},\; -\overline{A}_B x_B \leq -\underline{b},\; x_B \geq 0,\; x_N = 0.$$

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1}b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C1

- C1 says that $\mathbf{A}_B$ is regular;
- NP-hard problem;
- sufficient condition: $\rho\left(|((A_c)_B)^{-1}|(A_\Delta)_B\right) < 1$.

Basis stability

Non-interval case

Basis B is optimal iff

C1. A_B is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C2

- C2 says that the solution set to $\mathbf{A}_B x_B = \mathbf{b}$ lies in $\mathbb{R}^n_+$;
- sufficient condition: check of some enclosure to $\mathbf{A}_B x_B = \mathbf{b}$.

Basis stability

Non-interval case

Basis B is optimal iff

- C1. A_B is non-singular;
- C2. $A_B^{-1} b \geq 0$;
- C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Condition C3

- C2 says that $\mathbf{A}_N^T y \leq \mathbf{c}_N,\ \mathbf{A}_B^T y = \mathbf{c}_B$ is strongly feasible;
- NP-hard problem;
- sufficient condition:
  $\overline{(\mathbf{A}_N^T)\mathbf{y}} \leq \underline{c}_N$, where $\mathbf{y}$ is an enclosure to $\mathbf{A}_B^T y = \mathbf{c}_B$.

Basis stability

Theorem

Condition C3 holds true if and only if for each $q \in \{\pm 1\}^m$ the polyhedral set described by

$$
\begin{aligned}
((A_c)_B^T - (A_\Delta)_B^T \operatorname{diag}(q))y &\le \overline{c}_B,\\
-((A_c)_B^T + (A_\Delta)_B^T \operatorname{diag}(q))y &\le -\underline{c}_B,\\
\operatorname{diag}(q)\, y &\ge 0
\end{aligned}
$$

lies inside the polyhedral set

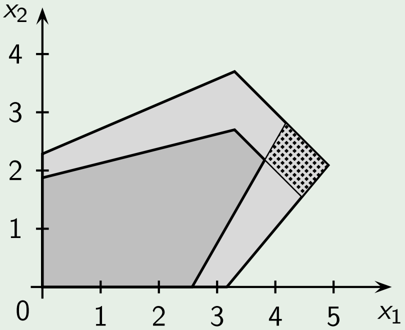
$$((A_c)_N^T + (A_\Delta)_N^T \operatorname{diag}(q))y \le \underline{c}_N,\ \operatorname{diag}(q)\, y \ge 0.$$

Basis stability

Example

Consider an interval linear program

$$\max\ ([5,6],[1,2])^T x \ \text{ s.t. } \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix},\ x \geq 0.$$



- union of all feasible sets in light gray,
- intersection of all feasible sets in dark gray,
- set of optimal solutions in dotted area

Interval linear programming problems

Open problems

- A sufficient and necessary condition for weak unboundedness, strong boundedness and weak optimality.
- A method to check if a given $x^* \in \mathbb{R}^n$ is an optimal solution for some realization.
- A method for determining the image of the optimal value function.
- A sufficient and necessary condition for duality gap to be zero for each realization.
- A method to test if a basis B is optimal for some realization.
- Tight enclosure to the optimal solution set.

Applications

Applications

Applications

- real-life problems affected by uncertainties
  - economics (portfolio selection,. . .)
  - environmental management (water resource and waste mng. planning)
  - logistic
  - . . .
- technical tool in constraint programming and global optimization
  technical tool in constraint programming and global optimization
- others
  - interval matrix games
  - interval linear regression interval linear regression
  - measure of sensitivity of linear programs

Interval linear regression

Linear regression

Consider a linear regression model

$$X\beta \approx y,$$

Find β solving

$$\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_p.$$

L_p-norm

- $\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_2$ . . . least squares

$$\beta = (X^TX)^{-1}X^Ty,$$

- $\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_1$ . . . least absolute deviations

$$\min e^Tw \ \text{ subject to } \ X\beta - y \le w,\ -X\beta + y \le w,\ w \ge 0.$$

- $\min_{\beta\in\mathbb{R}^m} \|X\beta - y\|_\infty$ . . . Chebyshev approximation

$$\min t \ \text{ subject to } \ X\beta - y \le te,\ -X\beta + y \le te,\ t \ge 0,$$

Interval linear regression

Interval linear regression

Consider a system of linear regression models

$$X\beta \approx y,$$

where $X \in \mathbf{X}$ and $y \in \mathbf{y}$.

Reduction to Interval linear programming

- For L_1-norm and L_∞-norm, we get an interval linear program.
- Optimal value range … minimal/maximal residual value
- Optimal solution set … set of all regression parameters
- Illustration of basis stability:
  - interpret β as a classifier of data (X, y) to two classes *below* and *above* regression line
  - basis stability = the same classification for any realization

Constraint programming and global optimization

Constraint programming problem

Enclose the set $\mathcal{S}$ described by

$$f_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, m, \qquad\qquad (\, f(x) = 0\,)$$
$$g_j(x_1, \ldots, x_n) \leq 0, \quad j = 1, \ldots, \ell, \qquad\qquad (\, g(x) \leq 0\,)$$

on a box $\mathbf{x}$.

Global optimization problem

Find

$$\min \varphi(x)$$

subject to

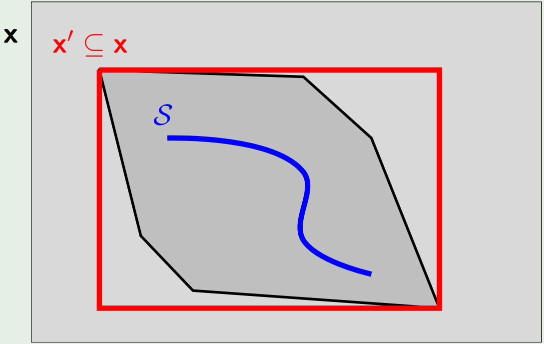
$$f(x) = 0, \quad g(x) \leq 0, \quad x \in \mathbf{x}.$$

Constraint programming and global optimization

Interval linear programming approach

- linearize constraints,
- compute new bounds and iterate.

Example



Constraint programming and global optimization

Interval linearization

Let $x^0 \in \mathbf{x}$. Suppose that a for some interval matrices $\mathbf{A}$ and $\mathbf{B}$ we have

$$f(x) \subseteq \mathbf{A}(x - x^0) + f(x^0), \quad \forall x \in \mathbf{x}$$
$$g(x) \subseteq \mathbf{B}(x - x^0) + g(x^0), \quad \forall x \in \mathbf{x},$$

e.g. by the mean value form, slopes, ...

Interval linear programming formulation

Now, the set $\mathcal{S}$ is enclosed by

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \leq 0.$$

What remains to do

- Solve the interval linear program
- choose $x^0 \in \mathbf{x}$

Case $x^0 := \underline{x}$

Let $x^0 := \underline{x}$. Since $x - \underline{x}$ is non-negative, the solution set to

$$\mathbf{A}(x - x^0) + f(x^0) = 0,$$
$$\mathbf{B}(x - x^0) + g(x^0) \le 0,$$

is described by

$$\underline{A}x \le \underline{A}\,\underline{x} - f(\underline{x}), \quad \overline{A}x \ge \overline{A}\underline{x} - f(\underline{x}),$$
$$\underline{B}x \le \underline{B}\,\underline{x} - g(\underline{x}).$$

- Similarly if x^0 is any other vertex of $\mathbf{x}$

Constraint programming and global optimization

General case

Let $x^0 \in \mathbf{x}$. The solution set to

$$\begin{aligned} \mathbf{A}(x - x^0) + f(x^0) &= 0, \\ \mathbf{B}(x - x^0) + g(x^0) &\leq 0, \end{aligned}$$

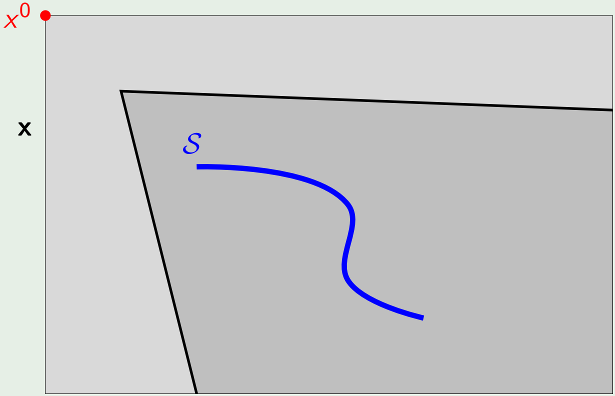
is described by

$$\begin{aligned} |A_c(x - x^0) + f(x^0)| &\leq A_\Delta |x - x^0|, \\ B_c(x - x^0) + g(x^0) \ &\leq B_\Delta |x - x^0|. \end{aligned}$$

- Non-linear description due to the absolute values.
- How to get rid of them?

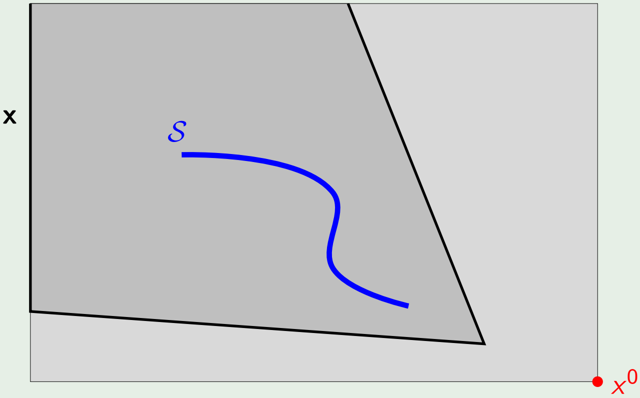
Example

Typical situation when choosing x^0 to be vertex:



Example

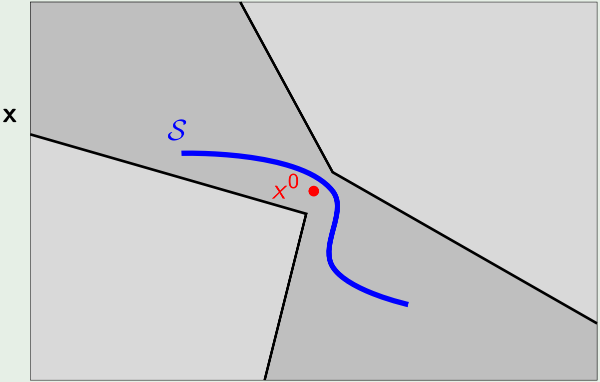
Typical situation when choosing x^0 to be the opposite vertex:



Constraint programming and global optimization

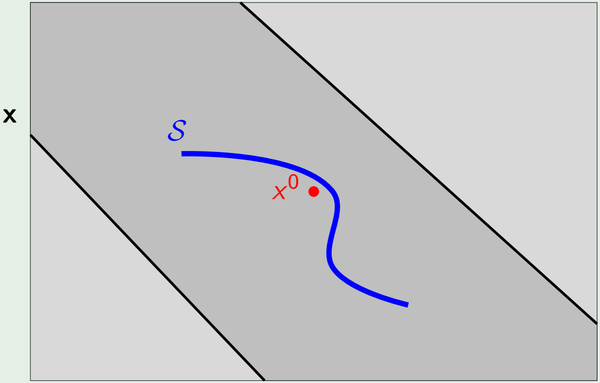
Example

Typical situation when choosing $x^0 = x_c$:



Example

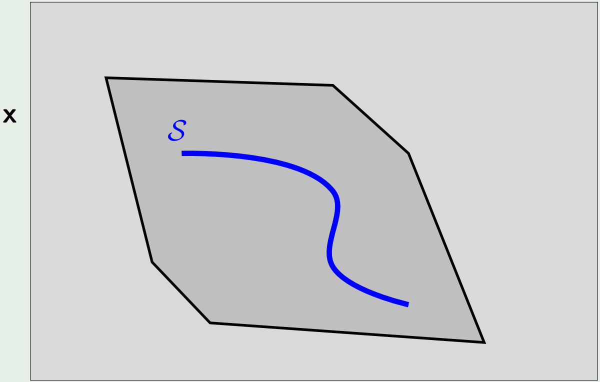
Typical situation when choosing $x^0 = x_c$ (after linearization):



Constraint programming and global optimization

Example

Typical situation when choosing all of them:



Conclusion

My apologies for not mentioning

- duality in interval linear programming
- linear programming verification
- fuzzy linear programming
- . . . and many others

Challenging problems

- enclose optimal solution set
- handle dependencies
- others (inner enclosures, . . .)