An interval linear programming contractor

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Motivation

Interval data are used to model:

- real life uncertainties
- measurement errors
- sensitivity analysis

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Introduction

Interval linear programming

Consider a linear programming problem

$$\min c^T x \;\text{ subject to }\; Ax = b,\; x \geq 0,$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

State of the art

- optimal value range (Chinneck & Ramadan, 2000, Hladík, 2009, Jansson, 2004, Mráz, 1998, Rohn, 2006, etc.)
- duality (Gabrel & Murat, 2010, Rohn, 1980, Serafini, 2005)
- basis stability (Beeck, 1978, Koníčková, 2001, Hladík, 2010, Rohn, 1993)
- optimal solution set (Beeck, 1978, Jansson, 1988, Machost, 1970)

Problem statement

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \mathbf{A},\, b \in \mathbf{b},\, c \in \mathbf{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Approaches

- Interval arithmetic (conservative)

Our approach

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, and $c \in \mathbf{c}$ such that

$$Ax = b,\ x \geq 0,\ A^T y \leq c,\ c^T x = b^T y$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Relaxation

$$Ax = b,\ x \geq 0,\ A'^T y \leq c,\ c'^T x = b'^T y,$$

where $A, A' \in \mathbf{A}$, $b, b' \in \mathbf{b}$, $c, c' \in \mathbf{c}$.

Our approach

Description of the relaxed problem

$$\begin{aligned} \underline{A}x &\le \overline{b}, \\ -\overline{A}x &\le -\underline{b}, \\ x &\ge 0, \\ A_c^T y - A_\Delta^T |y| &\le \overline{c}, \\ |c_c^T x - b_c^T y| &\le c_\Delta^T x + b_\Delta^T |y|. \end{aligned}$$

Properties

- The solution set is non-convex in general
- It is linear at any orthant
- NP-hard to obtain exact bounds

Idea

Linearize $|y|$.

Linearization of $|y|$

Theorem (Beaumont, 1998)

For every $y \in \mathbf{y} \subset \mathbb{R}$ with $\underline{y} < \overline{y}$ one has

$$|y| \leq \alpha y + \beta, \tag{1}$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \text{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \geq 0$ or $\overline{y} \leq 0$ then (1) holds as equation.

Now, the linearization reads

$$\begin{aligned} \underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,\\ \big(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)\big)y \le \overline{c} + A_\Delta^T\beta,\\ \underline{c}^T x + \big(-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T\beta,\\ -\overline{c}^T x + \big(b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T\beta, \end{aligned}$$

where

$$\alpha_i := \begin{cases} \frac{|\overline{y}_i|-|\underline{y}_i|}{\overline{y}_i-\underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i,\\ \operatorname{sgn}(\overline{y}_i) & \text{if } \underline{y}_i = \overline{y}_i, \end{cases}$$

$$\beta_i := \begin{cases} \frac{\overline{y}_i|\underline{y}_i|-\underline{y}_i|\overline{y}_i|}{\overline{y}_i-\underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i,\\ 0 & \text{if } \underline{y}_i = \overline{y}_i. \end{cases}$$

Contractor

Algorithm (Optimal solution set contractor)

1. Compute an initial interval enclosure $\mathbf{x}^0, \mathbf{y}^0$
2. $i := 0$;
3. **repeat**
    1. compute the interval hull $\mathbf{x}^i, \mathbf{y}^i$ of the linearized system;
    2. $i := i + 1$;
4. **until** improvement is nonsignificant;
5. **return** $\mathbf{x}^i$;

Properties

- Each iteration requires solving an interval hull ($2n$ linear programs).
- In practice, it converges quickly, but not to $\mathcal{S}$ in general.

Problems

- How to determine an initial enclosure $\mathbf{x}^0, \mathbf{y}^0$?

Example

Example

Consider an interval linear program

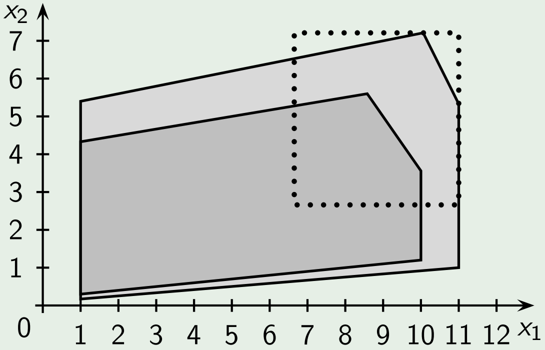
$$
\begin{aligned}
\min -[15,16]x_1 - [17,18]x_2 \quad & \text{subject to} \\
x_1 &\le [10,11], \\
-x_1 + [5,6]x_2 &\le [25,26], \\
[6,6.5]x_1 + [3,4.5]x_2 &\le [81,82], \\
-x_1 &\le -1, \\
x_1 - [10,12]x_2 &\le -[1,2].
\end{aligned}
$$

Take the initial enclosure

$$
\begin{aligned}
\mathbf{x}^0 &= 1000 \cdot ([-1,1],\ [-1,1])^T, \\
\mathbf{y}^0 &= 1000 \cdot ([0,1],\ [0,1],\ [0,1],\ [0,1],\ [0,1])^T.
\end{aligned}
$$

Example

Example (cont.)



- In grey the largest and the smallest feasible area.
- The final enclosure of the optimal solution set $\mathcal{S}$ is dotted.

Application: portfolio selection

Given:

- J possible investments;
- T time periods;
- r_{jt}, return on investment j in time period t;
- μ, risk aversion parameter (upper bound for risk).

Then:

- Estimated reward on investment j: $R_j := \frac{1}{T}\sum_{t=1}^{T} r_{jt}$;
- Risk measure of investment j: $\frac{1}{T}\sum_{t=1}^{T} |r_{jt} - R_j|$;
- Maximal allowed risk: $\frac{1}{T}\sum_{t=1}^{T} \left|\sum_{j=1}^{J}(r_{jt} - R_j)x_j\right| \leq \mu$.

Application: portfolio selection

Portfolio selection problem formulation

$$\max \sum_{j=1}^{J} R_j x_j$$

$$\text{subject to } -y_j \leq \sum_{j=1}^{J} (r_{jt} - R_j) x_j \leq y_t, \quad \forall t = 1, \ldots, T,$$

$$\sum_{j=1}^{J} x_j = 1, \; \frac{1}{T} \sum_{t=1}^{T} y_t \leq \mu,$$

$$x_j \geq 0, \quad \forall j = 1, \ldots, J,$$

where

$$R_j := \frac{1}{T} \sum_{t=1}^{T} r_{jt}.$$

Example

Example

$J = 4$ investments, $T = 5$ time periods, $\mu = 2$ risk aversion parameter.

The returns:

time period t	return on investment			
	1	2	3	4
1	10	20	9	11
2	12	25	11	14
3	9	17	12	12
4	11	21	11	14
5	11	19	13	16

- The optimal return: 20.08.
- The optimal solution: $x^* = (0, 0.9643, 0.0357, 0)^T$

Example

Suppose that the returns are not known precisely:

1. (1% tolerance) extending r_{tj} to intervals $[0.99 r_{tj}, 1.01 r_{tj}]$, we calculate:

$$\mathbf{x}^{(1)} = ([0, 0.1699],\ [0.7621, 1],\ [0, 0.181],\ [0, 0.2379])^T.$$

2. (5% tolerance) replace r_{jt} by intervals $[0.95 r_{tj}, 1.05 r_{tj}]$, $j \neq 2$

$$\mathbf{x}^{(2)} = ([0, 0.0495],\ [0.9276, 0.9712],\ [0, 0.0531],\ [0, 0.0724])^T.$$

Conclusion and future work

Conclusion

- Efective contractor for the optimal solution set $\mathcal{S}$.
- Each iteration requires solving $2n$ linear programs.
- In practice, it converges quickly.

Future work

- Initial enclosure of the optimal solution set $\mathcal{S}$.