Bounds on eigenvalues of complex interval matrices

Milan Hladík

Department of Applied Mathematics,
Faculty of Mathematics and Physics,
Charles University in Prague, Czech Republic

SWIM 2011, Bourges, France
June 14–15

Introduction

Interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n} \mid \underline{A} \le A \le \overline{A}\},$$

The corresponding center and radius matrices

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Complex interval matrix

$$\mathbf{A} + i\mathbf{B}.$$

Eigenvalue set

$$\Lambda(\mathbf{A} + i\mathbf{B}) = \{\lambda + i\mu \mid \exists A \in \mathbf{A}\ \exists B \in \mathbf{B}\ \exists x + iy \ne 0 : \\ (A + iB)(x + iy) = (\lambda + i\mu)(x + iy)\}.$$

Introduction

Selected publications

- **Deif, 1991**:
  exact bounds under strong assumptions
- **Mayer, 1994**:
  enclosure for the complex case based on Taylor expansion
- **Kolev and Petrakieva, 2005**:
  enclosure for real parts by solving nonlinear system of equations;
- **Rohn, 1998**:
  a cheap formula for an enclosure,
- **Hertz, 2009**:
  extension of Rohn's formula to the complex

Other directions

- approximations
- Hurwitz / Schur stability checking

Introduction

Notation

- $\rho(\cdot)$ spectral radius
- $\lambda_1(\cdot) \geq \cdots \geq \lambda_n(\cdot)$ eigenvalues of a symmetric matrix

Theorem (Rohn, 1998)

For each eigenvalue $\lambda + i\mu \in \Lambda(\mathbf{A})$ *we have*

$$\lambda \leq \lambda_1\left(\frac{1}{2}(A_c + A_c^T)\right) + \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),$$

$$\lambda \geq \lambda_n\left(\frac{1}{2}(A_c + A_c^T)\right) - \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),$$

$$\mu \leq \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} + \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix},$$

$$\mu \geq \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} - \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix}.$$

Introduction

Theorem (Hertz, 2009)

For each eigenvalue $\lambda + i\mu \in \Lambda(\mathbf{A} + i\mathbf{B})$ *we have*

$$\begin{aligned}
\lambda \leq\ & \lambda_1\left(\frac{1}{2}(A_c + A_c^T)\right) + \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right) \\
& + \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & 0 \end{pmatrix} + \rho\begin{pmatrix} 0 & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & 0 \end{pmatrix}, \\
\lambda \geq\ & \lambda_n\left(\frac{1}{2}(A_c + A_c^T)\right) - \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right) \\
& + \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & 0 \end{pmatrix} - \rho\begin{pmatrix} 0 & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & 0 \end{pmatrix}, \\
\mu \leq\ & \lambda_1\left(\frac{1}{2}(B_c + B_c^T)\right) + \rho\left(\frac{1}{2}(B_\Delta + B_\Delta^T)\right) \\
& + \lambda_1\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} + \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix}, \\
\mu \geq\ & \lambda_n\left(\frac{1}{2}(B_c + B_c^T)\right) - \rho\left(\frac{1}{2}(B_\Delta + B_\Delta^T)\right) \\
& + \lambda_n\begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} - \rho\begin{pmatrix} 0 & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & 0 \end{pmatrix}.
\end{aligned}$$

Symmetric case

Our approach

$\longrightarrow$ Reduction to the symmetric case.

Symmetric interval matrix

$$\mathbf{M}^S = \{M \in \mathbf{M} \mid M = M^T\}.$$

Eigenvalue sets

Let

$$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M).$$

be eigenvalues of a symmetric $M \in \mathbb{R}^{n \times n}$. Then

$$\boldsymbol{\lambda}_i(\mathbf{M}^S) = [\underline{\lambda}_i(\mathbf{M}^S), \overline{\lambda}_i(\mathbf{M}^S)] := \left\{\lambda_i(M) \mid M \in \mathbf{M}^S\right\}, \quad i = 1, \ldots, n.$$

Enclosures for the symmetric case

Theorem (Rohn, 2005)

$$\boldsymbol{\lambda}_i(\mathbf{A}^S) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)], \quad i = 1, \dots, n.$$

Theorem (Hertz, 1992)

Define $Z := \{1\} \times \{\pm 1\}^{n-1} = \{(1, \pm 1, \dots, \pm 1)\}$ *and for a* $z \in Z$ *define* $A_z, A'_z \in \mathbf{A}^S$ *in this way:*

$$(a_z)_{ij} = \begin{cases} \overline{a}_{ij} & \textit{if } s_i = s_j, \\ \underline{a}_{ij} & \textit{if } s_i \neq s_j, \end{cases}, \quad (a'_z)_{ij} = \begin{cases} \underline{a}_{ij} & \textit{if } s_i = s_j, \\ \overline{a}_{ij} & \textit{if } s_i \neq s_j. \end{cases}$$

Then

$$\overline{\lambda}_1(\mathbf{A}^S) = \max_{z \in Z} \lambda_1(A_z), \quad \underline{\lambda}_n(\mathbf{A}^S) = \min_{z \in Z} \lambda_n(A'_z).$$

Others

- Hladík, Daney and Tsigaridas, 2010, 2011

Main result

Theorem

For each eigenvalue $\lambda + i\mu \in \Lambda(\mathbf{A} + i\mathbf{B})$ *we have*

$$\underline{\lambda}_n \begin{pmatrix} \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) & \frac{1}{2}(\mathbf{B}^T - \mathbf{B}) \\ \frac{1}{2}(\mathbf{B} - \mathbf{B}^T) & \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \end{pmatrix}^S \leq \lambda \leq \overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) & \frac{1}{2}(\mathbf{B}^T - \mathbf{B}) \\ \frac{1}{2}(\mathbf{B} - \mathbf{B}^T) & \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \end{pmatrix}^S,$$

$$\underline{\lambda}_n \begin{pmatrix} \frac{1}{2}(\mathbf{B} + \mathbf{B}^T) & \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \\ \frac{1}{2}(\mathbf{A}^T - \mathbf{A}) & \frac{1}{2}(\mathbf{B} + \mathbf{B}^T) \end{pmatrix}^S \leq \mu \leq \overline{\lambda}_1 \begin{pmatrix} \frac{1}{2}(\mathbf{B} + \mathbf{B}^T) & \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \\ \frac{1}{2}(\mathbf{A}^T - \mathbf{A}) & \frac{1}{2}(\mathbf{B} + \mathbf{B}^T) \end{pmatrix}^S.$$

Main result

Corollary

For each $\lambda + i\mu \in \Lambda(\mathbf{A} + i\mathbf{B})$ *we have*

$$\lambda \le \lambda_1 \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{2}(A_\Delta + A_\Delta^T) & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{pmatrix},$$

$$\lambda \ge \lambda_n \begin{pmatrix} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(B_c^T - B_c) \\ \frac{1}{2}(B_c - B_c^T) & \frac{1}{2}(A_c + A_c^T) \end{pmatrix} - \rho \begin{pmatrix} \frac{1}{2}(A_\Delta + A_\Delta^T) & \frac{1}{2}(B_\Delta^T + B_\Delta) \\ \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{pmatrix},$$

$$\mu \le \lambda_1 \begin{pmatrix} \frac{1}{2}(B_c + B_c^T) & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & \frac{1}{2}(B_c + B_c^T) \end{pmatrix} + \rho \begin{pmatrix} \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & \frac{1}{2}(B_\Delta + B_\Delta^T) \end{pmatrix},$$

$$\mu \ge \lambda_n \begin{pmatrix} \frac{1}{2}(B_c + B_c^T) & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & \frac{1}{2}(B_c + B_c^T) \end{pmatrix} - \rho \begin{pmatrix} \frac{1}{2}(B_\Delta + B_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \\ \frac{1}{2}(A_\Delta^T + A_\Delta) & \frac{1}{2}(B_\Delta + B_\Delta^T) \end{pmatrix}.$$

Others

- using simple bounds for the symmetric case, but:
- for $\mathbf{B} = 0$ we have the same bounds as Rohn, 1998,
- in general, the bouds are as good as Hertz, 2009

Examples

Example (Seyranian, Kirillov, and Mailybaev, 2005)

Let

$$B(s) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} + i \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 2 \\ 4 & 2 & 0 \end{pmatrix} + 4i \begin{pmatrix} 0 & -s_1 - i\, s_2 & i\, s_3 \\ s_1 + i\, s_2 & 0 & -s_3 \\ -i\, s_3 & s_3 & 0 \end{pmatrix},$$

where $s \in \mathbf{s} = ([0, 0.2], [0.9797, 1], 0)$.

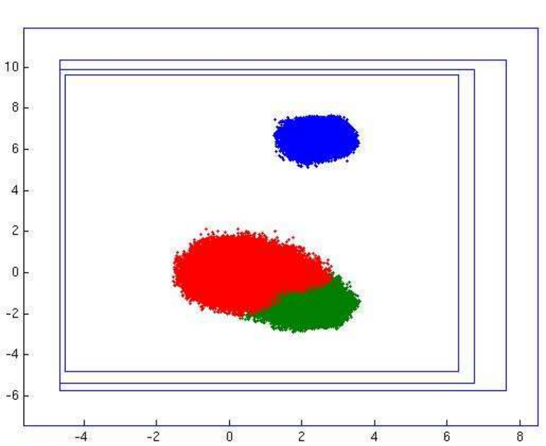
- Hertz enclosure: $\lambda \in [-5.6732, 8.5134]$, $\quad \mu \in [-7.4311, 11.8843]$.

Our approach

- simple bounds: $\lambda \in [-4.6546, 7.6146]$, $\quad \mu \in [-5.7632, 10.3387]$.
- by filtering: $\lambda \in [-4.6546, 6.7421]$, $\quad \mu \in [-5.4017, 9.8787]$.
- best bounds: $\lambda \in [-4.5180, 6.3031]$, $\quad \mu \in [-4.8237, 9.6227]$.

Example (con't)

Monter Carlo simulation:



Examples

Example (Petkovski, 1991)

Let

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & -1 \\ 2 & [-1.399, -0.001] & 0 \\ 1 & 0.5 & -1 \end{pmatrix}$$

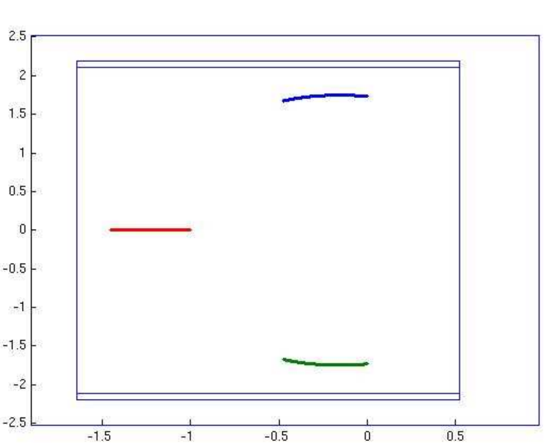
- Hertz enclosure: $\lambda \in [-1.9068, 0.9702], \quad \mu \in [-2.5191, 2.5191].$

Our approach

- simple bounds: $\lambda \in [-1.9068, 0.9702], \quad \mu \in [-2.5191, 2.5191].$
- by filtering: $\lambda \in [-1.6474, 0.5205], \quad \mu \in [-2.1934, 2.1934].$
- best bounds: $\lambda \in [-1.6474, 0.5205], \quad \mu \in [-2.1112, 2.1112].$

Example (con't)

Monter Carlo simulation:



Examples

Example (Xiao and Unbehauen, 2000)

Let

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & [-1,1] \\ 0 & -1 & [-1,1] \\ [-1,1] & [-1,1] & 0.1 \end{pmatrix}$$

- Hertz enclosure: $\lambda \in [-2.4143, 1.5143], \quad \mu \in [-1.4143, 1.4143].$

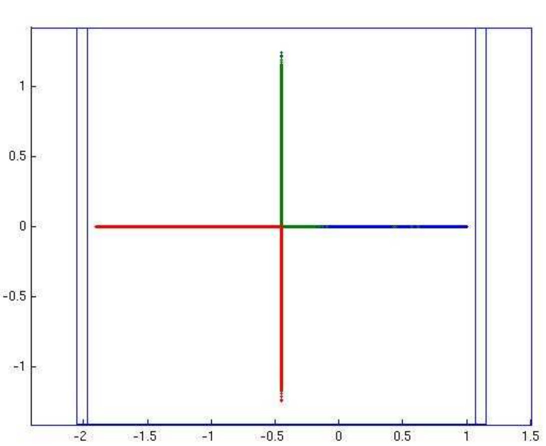
Our approach

- simple bounds: $\lambda \in [-2.4143, 1.5143], \quad \mu \in [-1.4143, 1.4143].$
- by filtering: $\lambda \in [-2.0532, 1.1532], \quad \mu \in [-1.4143, 1.4143].$
- best bounds: $\lambda \in [-1.9674, 1.0674], \quad \mu \in [-1.4143, 1.4143].$

Examples

Example (con't)

Monter Carlo simulation:



Examples

Example (Wang, Michel and Liu, 1994)

Let

$$\mathbf{A} = \begin{pmatrix} [-3,-2] & [4,5] & [4,6] & [-1,1.5] \\ [-4,-3] & [-4,-3] & [-4,-3] & [1,2] \\ [-5,-4] & [2,3] & [-5,-4] & [-1,0] \\ [-1,0.1] & [0,1] & [1,2] & [-4,2.5] \end{pmatrix}$$

- Hertz enclosure: $\lambda \in [-8.8221, 3.4408], \quad \mu \in [-10.7497, 10.7497]$.

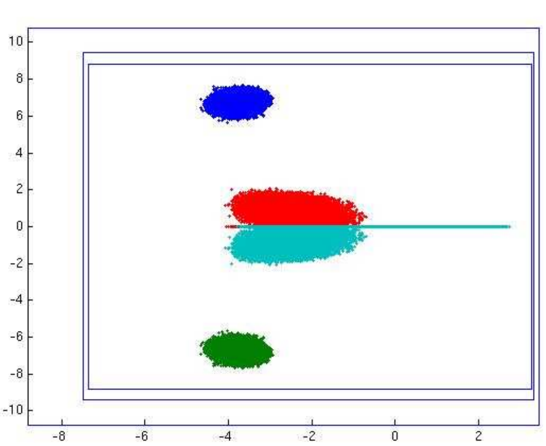
Our approach

- simple bounds: $\lambda \in [-8.8221, 3.4408], \quad \mu \in [-10.7497, 10.7497]$.
- by filtering: $\lambda \in [-7.4848, 3.3184], \quad \mu \in [-9.4224, 9.4224]$.
- best bounds: $\lambda \in [-7.3691, 3.2742], \quad \mu \in [-8.7948, 8.7948]$.

Examples

Example (con't)

Monter Carlo simulation:



Conclusion

Conclusion

- reduction of the problem to real symmetric one
- cheap and tight enclosure
- outperforms Rohn (1998) and Hertz (2009) formulae