An interval linear programming contractor

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OR Zurich 2011, Switzerland
August 30 – September 2

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Motivation

Interval data are used to model:

- real life uncertainties
- measurement errors
- sensitivity analysis

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Introduction

Interval linear programming

Consider a linear programming problem

$$\min c^T x \ \text{ subject to } \ Ax = b,\ x \geq 0,$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

State of the art

- optimal value range (Chinneck & Ramadan, 2000, Hladík, 2009, Jansson, 2004, Mráz, 1998, Rohn, 2006, etc.)
- duality (Gabrel & Murat, 2010, Rohn, 1980, Serafini, 2005)
- basis stability (Beeck, 1978, Koníčková, 2001, Hladík, 2010, Rohn, 1993)
- optimal solution set (Beeck, 1978, Jansson, 1988, Machost, 1970)

Problem statement

The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \mathbf{A},\, b \in \mathbf{b},\, c \in \mathbf{c}} \mathcal{S}(A, b, c).$$

Goal

Find a tight enclosure to $\mathcal{S}$.

Approaches

- Interval arithmetic (conservative)

Our approach

Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, and $c \in \mathbf{c}$ such that

$$Ax = b,\ x \geq 0,\ A^T y \leq c,\ c^T x = b^T y$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Relaxation

$$Ax = b,\ x \geq 0,\ {A'}^T y \leq c,\ {c'}^T x = {b'}^T y,$$

where $A, A' \in \mathbf{A}$, $b, b' \in \mathbf{b}$, $c, c' \in \mathbf{c}$.

Our approach

Description of the relaxed problem

$$\underline{A}x \le \overline{b},$$
$$-\overline{A}x \le -\underline{b},$$
$$x \ge 0,$$
$$A_c^T y - A_\Delta^T |y| \le \overline{c},$$
$$|c_c^T x - b_c^T y| \le c_\Delta^T x + b_\Delta^T |y|.$$

Properties

- The solution set is non-convex in general
- It is linear at any orthant
- NP-hard to obtain exact bounds

Idea

Linearize $|y|$.

Linearization of $|y|$

Theorem (Beaumont, 1998)

For every $y \in \mathbf{y} \subset \mathbb{R}$ with $\underline{y} < \overline{y}$ one has

$$|y| \leq \alpha y + \beta, \tag{1}$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \textit{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \geq 0$ or $\overline{y} \leq 0$ then (1) holds as equation.

Linearization of $|y|$

Now, the linearization reads

$$\begin{aligned} \underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0,\\ \big(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)\big)y \le \overline{c} + A_\Delta^T\beta,\\ \underline{c}^T x + \big(-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T\beta,\\ -\overline{c}^T x + \big(b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big)y \le b_\Delta^T\beta, \end{aligned}$$

where

$$\alpha_i := \begin{cases} \frac{|\overline{y}_i| - |\underline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i,\\ \operatorname{sgn}(\overline{y}_i) & \text{if } \underline{y}_i = \overline{y}_i, \end{cases}$$

$$\beta_i := \begin{cases} \frac{\overline{y}_i|\underline{y}_i| - \underline{y}_i|\overline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i,\\ 0 & \text{if } \underline{y}_i = \overline{y}_i. \end{cases}$$

Contractor

Algorithm (Optimal solution set contractor)

1. Compute an initial interval enclosure $\mathbf{x}^0, \mathbf{y}^0$
2. $i := 0$;
3. **repeat**
   1. compute the interval hull $\mathbf{x}^i, \mathbf{y}^i$ of the linearized system;
   2. $i := i + 1$;
4. **until** improvement is nonsignificant;
5. **return** $\mathbf{x}^i$;

Properties

- Each iteration requires solving an interval hull ($2n$ linear programs).
- In practice, it converges quickly, but not to $\mathcal{S}$ in general.

Problems

- How to determine an initial enclosure $\mathbf{x}^0, \mathbf{y}^0$?

Example

Example

Consider an interval linear program

$$\begin{aligned}\min -[15,16]x_1 - [17,18]x_2 \ \ \text{subject to}\\ x_1 \leq [10,11],\\ -x_1 + [5,6]x_2 \leq [25,26],\\ [6,6.5]x_1 + [3,4.5]x_2 \leq [81,82],\\ -x_1 \leq -1,\\ x_1 - [10,12]x_2 \leq -[1,2].\end{aligned}$$

Take the initial enclosure

$$\mathbf{x}^0 = 1000 \cdot ([-1,1],\ [-1,1])^T,$$
$$\mathbf{y}^0 = 1000 \cdot ([0,1],\ [0,1],\ [0,1],\ [0,1],\ [0,1])^T.$$

Example

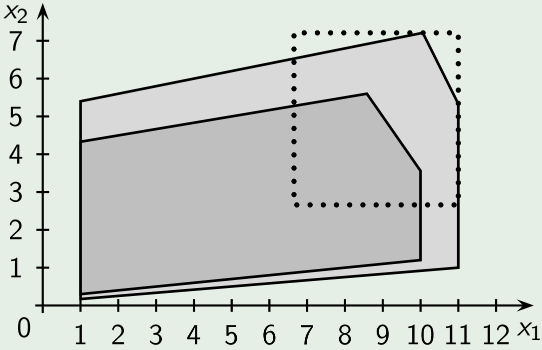
Example (cont.)

The iterations of the procedure go as follows

$$
\begin{aligned}
\mathbf{x}^0 &= 1000 \cdot ([-1,1],\ [-1,1])^T,\\
\mathbf{y}^0 &= 1000 \cdot ([0,1],\ [0,1],\ [0,1],\ [0,1],\ [0,1])^T,\\
\mathbf{x}^1 &= ([1,11],\ [-568,916])^T,\\
\mathbf{y}^1 &= ([0,1000],\ [0,936],\ [0,358],\ [0,1000],\ [0,572])^T,\\
\mathbf{x}^2 &= ([1,11],\ [-17.2,72])^T,\\
\mathbf{y}^2 &= ([0,190],\ [0,58.5],\ [0,24.3],\ [0,176],\ [0,34.6])^T,\\
\mathbf{x}^3 &= ([3.78,11],\ [1.91,9.80])^T,\\
\mathbf{y}^3 &= ([0,30.6],\ [0,6.98],\ [4.71],\ [0,17.1],\ [0,3.09])^T,\\
\mathbf{x}^4 &= ([6.65,11],\ [2.66,7.21])^T,\\
\mathbf{y}^4 &= ([0,22.5],\ [0.08,4.33],\ [0,3.67],\ [0,8.81],\ [0,1.47])^T.
\end{aligned}
$$

Example

Example (cont.)



- In grey the largest and the smallest feasible area.
- The final enclosure of the optimal solution set $\mathcal{S}$ is dotted.

Conclusion and future work

Conclusion

- Efective contractor for the optimal solution set $\mathcal{S}$.
- Each iteration requires solving $2n$ linear programs.
- In practice, it converges quickly.

Future work

- Initial enclosure of the optimal solution set $\mathcal{S}$.