Error bounds on the spectral radius of uncertain matrices

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Why intervals?

Use intervals

- to handle uncertainty
- for their simplicity – just lower and upper bound
- when dealing with continuum of states
- for sensitivity analysis
- for reliable results

State-of-the-art

Enclosures for eigenvalues of interval matrices

- Franzè, Carotenuto and Balestrino (2006):
  Gerschgorin-like regions for locating eigenvalues
- Mayer (1994):
  an enclosure method for eigenvalues based on Taylor expansion
- Ahn, Moore and Chen (2006):
  an estimation on eigenvalues based on perturbation theory
- Rohn (1998):
  a cheap formula for an eigenvalue enclosure
- H., Daney and Tsigaridas (2010):
  several formulae for eigenvalue enclosures

- Plenty of papers on checking Schur/Hurwitz stability of interval matrices

Notation

Interval matrices

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\}.$$

The center and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

A symmetric interval matrix

$$\mathbf{A}^S = \{A \in \mathbf{A} \mid A = A^T\}.$$

Notation

Eigenvalues

Eigenvalues of a real symmetric matrix $A \in \mathbb{R}^{n\times n}$:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Eigenvalues of a symmetric interval matrix $\mathbf{A}^S$:

$$\boldsymbol{\lambda}_i(\mathbf{A}^S) = [\underline{\lambda}_i(\mathbf{A}^S), \overline{\lambda}_i(\mathbf{A}^S)] := \left\{\lambda_i(A) \mid A \in \mathbf{A}^S\right\}, \quad i = 1, \dots, n,$$

The spectral radius of $\mathbf{A}$:

$$\rho(\mathbf{A}) = \max\{\rho(A);\ A \in \mathbf{A}\}.$$

The maximum singular value of $\mathbf{A}$:

$$\sigma(\mathbf{A}) = \max\{\sigma(A);\ A \in \mathbf{A}\}.$$

First error bound

Lemma

We have $\overline{\lambda}_1(\mathbf{A}^S) \leq \lambda_1(A_c) + \rho(A_\Delta)$.

Theorem

We have $\rho(\mathbf{A}) \leq \sigma(A_c) + \sigma(A_\Delta)$.

Proof.

Use the fact

$$\rho(A) \leq \sigma(A) = \lambda_1 \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}.$$

and Lemma. □

Second error bound

Theorem

Denote

$$\mathbf{Z}^S := \begin{pmatrix} \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) & \frac{1}{2}(\mathbf{A}-\mathbf{A}^T) \\ \frac{1}{2}(\mathbf{A}^T-\mathbf{A}) & \frac{1}{2}(\mathbf{A}+\mathbf{A}^T) \end{pmatrix}^S.$$

Then we have $\rho(\mathbf{A}) \leq \rho(\mathbf{Z}^S)$.

Corollary

We have $\rho(\mathbf{A}) \leq \rho(Z_c) + \rho(Z_\Delta)$, *where*

$$Z_c = \begin{pmatrix} \frac{1}{2}(A_c+A_c^T) & \frac{1}{2}(A_c-A_c^T) \\ \frac{1}{2}(A_c^T-A_c) & \frac{1}{2}(A_c+A_c^T) \end{pmatrix}, \; Z_\Delta = \begin{pmatrix} \frac{1}{2}(A_\Delta+A_\Delta^T) & \frac{1}{2}(A'_\Delta+A'^T_\Delta) \\ \frac{1}{2}(A'_\Delta+A'^T_\Delta) & \frac{1}{2}(A_\Delta+A_\Delta^T) \end{pmatrix}$$

and A'_Δ *is defined as* $(A'_\Delta)_{ij} = (A_\Delta)_{ij}$ *if* $i \neq j$ *and* $(A'_\Delta)_{ij} = 0$ *otherwise.*

Refinement

Theorem (H., Daney and Tsigaridas, 2011)

Suppose that ρ is an eigenvalue of no matrix in $\mathbf{B}^S$ and put $\mathbf{M}^S := \mathbf{B}^S - \rho I$. Then $\rho + \lambda$ is an eigenvalue of no matrix in $\mathbf{B}^S$ for all real λ satisfying

$$|\lambda| < \frac{2 - \rho\left(|I - QM_c| + |I - M_cQ| + |Q|M_\Delta + M_\Delta|Q|\right)}{2\rho(|Q|)},$$

where $Q \in \mathbb{R}^{n\times n}$, $Q \neq 0$, is an arbitrary symmetric matrix.

For the first error bound, we put:

$$\rho := \sigma(A_c) + \sigma(A_\Delta), \quad \mathbf{B}^S := \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix}^S.$$

For the second error bound, we use it possibly twice:

$$\rho := \lambda_1(\pm Z_c) + \rho(Z_\Delta), \quad \mathbf{B}^S := \pm \mathbf{Z}^S.$$

Refinement

Algorithm

compute an initial upper bound $\rho \geq \overline{\lambda}_1(\mathbf{B}^S)$;
$t := 0; \quad \lambda := \varepsilon + 1$;
while $\lambda > \varepsilon$ and $t < T$ **do**
    $t := t + 1$;
    $\mathbf{M}^S := \mathbf{B}^S - \rho I$;
    compute approximately $Q \approx M_c^{-1}$;
    $\lambda := \dfrac{2 - \rho\left(|I - QM_c| + |I - M_c Q| + |Q|M_\Delta + M_\Delta|Q|\right)}{2\rho\left(|Q|\right)}$;
    if $\lambda > 0$ **then**
        $\rho := \rho - \lambda$;
    end if
end while
return ρ.

Example 1

Example

Consider an example from Franzè, Carotenuto and Balestrino (2006)

$$\mathbf{A} = \begin{pmatrix} [-0.2, 0.16] & [-0.34, 0.02] \\ [-0.24, 0.12] & [-0.16, 0.2] \end{pmatrix}.$$

The first bound is $\rho(\mathbf{A}) \leq \rho = 0.5219$.

Algorithm refines the upper bound on the spectral radius as follows

$$\rho = 0.5219 \ \rightarrow \ 0.4953 \ \rightarrow \ 0.4936 \ \rightarrow \ 0.4934.$$

The stability margin is $S \geq 1 - 0.4934 = 0.5066$, which is higher value than 0.144 reported.

The second approach yields $\rho(\mathbf{A}) \leq 0.6625$. Algorithm makes no improvement.

Example 2

Example

Consider an example from Ghosh, Sen and Datta (2000)

$$\mathbf{A} = \begin{pmatrix} [-0.4, 0.72] & 0.3 \\ [-0.45, 0.6] & -0.15 \end{pmatrix}.$$

The first bound reads $\rho(\mathbf{A}) \leq \rho = 1.1235$.

Algorithm yields a decreasing sequence

$$\rho = 1.1235 \ \rightarrow \ 1.0029 \ \rightarrow \ 0.9868 \ \rightarrow \ 0.9844$$

terminating below one. Thus, $\rho(\mathbf{A}) < 1$ and the matrix is Schur stable.

The second approach gives $\rho(\mathbf{A}) \leq 1.1480$, which is not refined by Algorithm.

Example 3

Example

Consider an interval matrix

$$\mathbf{A} = \begin{pmatrix} [4.4, 4.6] & [-3.5, -3.3] & [0.9, 1.1] & [4.3, 4.5] & [-4.8, -4.6] \\ [-0.6, -0.4] & [2.6, 2.8] & [-0.9, -0.7] & [3.1, 3.3] & [-5, -4.8] \\ [1, 1.2] & [2.3, 2.5] & [-2.8, -2.6] & [-4.4, -4.2] & [4.8, 5] \\ [-1.9, -1.7] & [-0.1, 0.1] & [-1.3, -1.1] & [-1, -0.8] & [-5, -4.8] \\ [-2.6, -2.4] & [3.7, 3.9] & [-2.7, -2.5] & [-1.3, -1.1] & [-2.5, -2.3] \end{pmatrix}.$$

The first approach results in the error bound $\rho(\mathbf{A}) \leq 12.2417$.

Algorithm refines it to 12.1374.

But the second error bound $\rho(\mathbf{A}) \leq 10.7796$ is more accurate.

Conclusion

Summary

- two cheap formulae for the spectral radius of interval matrices
- refinement
- use as is or as a starting point for other algorithms

Future directions

- enclosure for complex eigenvalues of interval matrices

References

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