Error bounds on the spectral radius of uncertain matrices

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Why intervals?

Use intervals

- to handle uncertainty
- for their simplicity – just lower and upper bound
- when dealing with continuum of states
- for sensitivity analysis
- for reliable results
State-of-the-art

Enclosures for eigenvalues of interval matrices

- Franzè, Carotenuto and Balestrino (2006): Gerschgorin-like regions for locating eigenvalues
- Mayer (1994): an enclosure method for eigenvalues based on Taylor expansion
- Ahn, Moore and Chen (2006): an estimation on eigenvalues based on perturbation theory
- Rohn (1998): a cheap formula for an eigenvalue enclosure
- H., Daney and Tsingaridas (2010): several formulae for eigenvalue enclosures

Plenty of papers on checking Schur/Hurwitz stability of interval matrices
Interval matrices

An interval matrix

\[ A := [A, \overline{A}] = \{ A \in \mathbb{R}^{m \times n} \mid A \leq A \leq \overline{A} \}. \]

The center and radius matrices

\[ A_c := \frac{1}{2}(\overline{A} + A), \quad A_\Delta := \frac{1}{2}(\overline{A} - A). \]

A symmetric interval matrix

\[ A^S = \{ A \in A \mid A = A^T \}. \]
Eigenvalues of a real symmetric matrix $A \in \mathbb{R}^{n \times n}$:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Eigenvalues of a symmetric interval matrix $A^S$:

$$\lambda_i(A^S) = [\Delta_i(A^S), \overline{\lambda}_i(A^S)] := \{\lambda_i(A) | A \in A^S\}, \quad i = 1, \ldots, n,$$

The spectral radius of $A$:

$$\rho(A) = \max \{\rho(A); \ A \in A\}.$$

The maximum singular value of $A$:

$$\sigma(A) = \max \{\sigma(A); \ A \in A\}.$$
Lemma

We have $\overline{\lambda}_1(A^S) \leq \lambda_1(A_c) + \rho(A_\Delta)$.

Theorem

We have $\rho(A) \leq \sigma(A_c) + \sigma(A_\Delta)$.

Proof.

Use the fact

$$\rho(A) \leq \sigma(A) = \lambda_1 \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}.$$ 

and Lemma.
Second error bound

**Theorem**

Denote

\[ Z^S := \left( \begin{array}{cc} \frac{1}{2}(A + A^T) & \frac{1}{2}(A - A^T) \\ \frac{1}{2}(A^T - A) & \frac{1}{2}(A + A^T) \end{array} \right)^S. \]

Then we have \( \rho(A) \leq \rho(Z^S) \).

**Corollary**

We have \( \rho(A) \leq \rho(Z_c) + \rho(Z_\Delta) \), where

\[ Z_c = \left( \begin{array}{cc} \frac{1}{2}(A_c + A_c^T) & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & \frac{1}{2}(A_c + A_c^T) \end{array} \right), \quad Z_\Delta = \left( \begin{array}{cc} \frac{1}{2}(A_\Delta + A_\Delta^T) & \frac{1}{2}(A'_\Delta + A'_\Delta^T) \\ \frac{1}{2}(A'_\Delta + A'_\Delta^T) & \frac{1}{2}(A_\Delta + A_\Delta^T) \end{array} \right) \]

and \( A'_\Delta \) is defined as \( (A'_\Delta)_{ij} = (A_\Delta)_{ij} \) if \( i \neq j \) and \( (A'_\Delta)_{ij} = 0 \) otherwise.
Theorem (H., Daney and Tsiganidas, 2011)

Suppose that $\rho$ is an eigenvalue of no matrix in $\mathbf{B}^S$ and put $\mathbf{M}^S := \mathbf{B}^S - \rho \mathbf{I}$. Then $\rho + \lambda$ is an eigenvalue of no matrix in $\mathbf{B}^S$ for all real $\lambda$ satisfying

$$|\lambda| < \frac{2 - \rho (|I - QM_c| + |I - M_cQ| + |Q|M_\Delta + M_\Delta|Q|)}{2\rho(|Q|)},$$

where $Q \in \mathbb{R}^{n \times n}$, $Q \neq 0$, is an arbitrary symmetric matrix.

For the first error bound, we put:

$$\rho := \sigma(A_c) + \sigma(A_\Delta), \quad \mathbf{B}^S := \left( \begin{array}{cc} 0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{array} \right)^S.$$

For the second error bound, we use it possibly twice:

$$\rho := \lambda_1(\pm Z_c) + \rho(Z_\Delta), \quad \mathbf{B}^S := \pm \mathbf{Z}^S.$$
Refinement

Algorithm

1: compute an initial upper bound $\rho \geq \lambda_1(B^S)$;
2: $t := 0; \quad \lambda := \varepsilon + 1;$
3: while $\lambda > \varepsilon$ and $t < T$ do
4: \hspace{1em} $t := t + 1;$
5: \hspace{1em} $M^S := B^S - \rho I;$
6: \hspace{1em} compute approximately $Q \approx M^{-1}_c$;
7: \hspace{1em} $\lambda := \frac{2 - \rho (|I - QM_c| + |I - M_c Q| + |Q| M_\Delta + M_\Delta |Q|)}{2 \rho (|Q|)}$;
8: \hspace{1em} if $\lambda > 0$ then
9: \hspace{2em} $\rho := \rho - \lambda;$
10: \hspace{1em} end if
11: \hspace{1em} end while
12: return $\rho$. 
Example 1

Consider an example from Franzè, Carotenuto and Balestrino (2006)

\[ A = \begin{pmatrix}
-0.2 & 0.16 \\
-0.24 & 0.12 \\
-0.16 & 0.2
\end{pmatrix} \begin{pmatrix}
-0.34 & 0.02 \\
-0.16 & 0.2
\end{pmatrix}. \]

The first bound is \( \rho(A) \leq \rho = 0.5219 \).

Algorithm refines the upper bound on the spectral radius as follows

\[ \rho = 0.5219 \rightarrow 0.4953 \rightarrow 0.4936 \rightarrow 0.4934. \]

The stability margin is \( S \geq 1 - 0.4934 = 0.5066 \), which is higher value than 0.144 reported.

The second approach yields \( \rho(A) \leq 0.6625 \). Algorithm makes no improvement.
Example 2

Consider an example from Ghosh, Sen and Datta (2000)

\[
A = \begin{pmatrix}
-0.4 & 0.72 \\
-0.45 & 0.6
\end{pmatrix}
\begin{pmatrix}
0.3 \\
-0.15
\end{pmatrix}.
\]

The first bound reads \(\rho(A) \leq \rho = 1.1235\).

Algorithm yields a decreasing sequence

\[
\rho = 1.1235 \rightarrow 1.0029 \rightarrow 0.9868 \rightarrow 0.9844
\]

terminating below one. Thus, \(\rho(A) < 1\) and the matrix is Schur stable.

The second approach gives \(\rho(A) \leq 1.1480\), which is not refined by Algorithm.
Consider an interval matrix

\[
A = \begin{pmatrix}
[4.4, 4.6] & [-3.5, -3.3] & [0.9, 1.1] & [4.3, 4.5] & [-4.8, -4.6] \\
[-0.6, -0.4] & [2.6, 2.8] & [-0.9, -0.7] & [3.1, 3.3] & [-5, -4.8] \\
[-1.9, -1.7] & [-0.1, 0.1] & [-1.3, -1.1] & [-1, -0.8] & [-5, -4.8] \\
\end{pmatrix}
\]

The first approach results in the error bound \( \rho(A) \leq 12.2417 \).

Algorithm refines it to 12.1374.

But the second error bound \( \rho(A) \leq 10.7796 \) is more accurate.
Conclusion

Summary
- two cheap formulae for the spectral radius of interval matrices
- refinement
- use as is or as a starting point for other algorithms

Future directions
- enclosure for complex eigenvalues of interval matrices
References


