# Interval Linear Programming: Foundations, Tools and Challenges


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# Introduction

## Linear programming

Three basic forms of linear programs

$$\begin{aligned} f(A,b,c) &\equiv \min c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0, \\ f(A,b,c) &\equiv \min c^T x \ \text{ subject to } \ Ax \leq b, \\ f(A,b,c) &\equiv \min c^T x \ \text{ subject to } \ Ax \leq b, \ x \geq 0. \end{aligned}$$

## Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

# Introduction

## Interval linear programming

Family of linear programs with $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, in short

$$f(\mathbf{A}, \mathbf{b}, \mathbf{c}) \equiv \min \mathbf{c}^T x \;\text{ subject to }\; \mathbf{A}x \overset{(\leq)}{=} \mathbf{b},\; (x \geq 0).$$

A scenario is a concrete linear program in this family.

The three forms are not transformable between each other!

## Goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

# Complexity of basic problems

| | $\mathbf{A}x = \mathbf{b},\ x \geq 0$ | $\mathbf{A}x \leq \mathbf{b}$ | $\mathbf{A}x \leq \mathbf{b},\ x \geq 0$ |
|---|---|---|---|
| strong feasibility | NP-hard | polynomial | polynomial |
| weak feasibility | polynomial | NP-hard | polynomial |
| strong unboundedness | NP-hard | polynomial | polynomial |
| weak unboundedness | suff. / necessary conditions only | suff. / necessary conditions only | polynomial |
| strong optimality | NP-hard | NP-hard | polynomial |
| weak optimality | suff. / necessary conditions only | suff. / necessary conditions only | suff. / necessary conditions only |
| optimal value range | $\underline{f}$ polynomial<br>$\overline{f}$ NP-hard | $\underline{f}$ NP-hard<br>$\overline{f}$ polynomial | polynomial |

# Optimal value range

## Definition

$$\underline{f} := \min f(A, b, c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c},$$
$$\overline{f} := \max f(A, b, c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}.$$

## Theorem (Rohn, 2006)

*We have for type* $(\mathbf{A}x = \mathbf{b},\ x \geq 0)$

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0,$$
$$\overline{f} = \max_{p \in \{\pm 1\}^m} f(A_c - \mathrm{diag}(p)\, A_\Delta, b_c + \mathrm{diag}(p)\, b_\Delta, \overline{c}).$$

## Theorem (Vajda, 1961)

*We have for type* $(\mathbf{A}x \leq \mathbf{b},\ x \geq 0)$

$$\underline{f} = \min \underline{c}^T x \ \textit{ subject to } \ \underline{A}x \leq \overline{b},\ x \geq 0,$$
$$\overline{f} = \min \overline{c}^T x \ \textit{ subject to } \ \overline{A}x \leq \underline{b},\ x \geq 0.$$

**Algorithm (Optimal value range $[\underline{f}, \overline{f}]$)**

1. Compute

$$\underline{f} := \inf \ c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ x \in \mathcal{M},$$

   where $\mathcal{M}$ is the primal solution set.

2. If $\underline{f} = \infty$, then set $\overline{f} := \infty$ and stop.
3. Compute

$$\overline{\varphi} := \sup \ b_c^T y + b_\Delta^T |y| \ \text{ subject to } \ y \in \mathcal{N},$$

   where $\mathcal{N}$ is the dual solution set.

4. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
5. If the primal problem is strongly feasible, then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

# Optimal solution set

## The optimal solution set

Denote by $\mathcal{S}(A, b, c)$ the set of optimal solutions to

$$\min c^T x \;\text{ subject to }\; Ax = b,\; x \geq 0,$$

Then the optimal solution set is defined

$$\mathcal{S} := \bigcup_{A \in \mathbf{A},\, b \in \mathbf{b},\, c \in \mathbf{c}} \mathcal{S}(A, b, c).$$

## Goal

Find a tight enclosure to $\mathcal{S}$.

# Optimal solution set

## Characterization

By duality theory, we have that $x \in \mathcal{S}$ if and only if there is some $y \in \mathbb{R}^m$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, and $c \in \mathbf{c}$ such that

$$Ax = b,\ x \geq 0,\ A^T y \leq c,\ c^T x = b^T y,$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

## Relaxation

Relaxing the dependencies

$$\mathbf{A}x = \mathbf{b},\ x \geq 0,\ \mathbf{A}^T y \leq \mathbf{c},\ \mathbf{c}^T x = \mathbf{b}^T y,$$

which is described by

$$\begin{gathered} \underline{A}x \leq \overline{b}, \quad -\overline{A}x \leq -\underline{b}, \quad x \geq 0, \\ A_c^T y - A_\Delta^T |y| \leq \overline{c}, \quad |c_c^T x - b_c^T y| \leq c_\Delta^T x + b_\Delta^T |y|. \end{gathered}$$

# Linearization of $|y|$

## Properties

- The solution set is non-convex in general
- It is linear at any orthant
- NP-hard to obtain exact bounds

## Theorem (Beaumont, 1998)

*For every* $y \in \mathbf{y} \subset \mathbb{R}$ *with* $\underline{y} < \overline{y}$ *one has*

$$|y| \leq \alpha y + \beta, \tag{1}$$

*where*

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}} \quad \text{and} \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

*Moreover, if* $\underline{y} \geq 0$ *or* $\overline{y} \leq 0$ *then (1) holds as equation.*

# Linearization of $|y|$

Now, the linearization reads

$$\begin{aligned} \underline{A}x \le \overline{b},\ -\overline{A}x \le -\underline{b},\ x \ge 0, \\ \big(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)\big) y \le \overline{c} + A_\Delta^T \beta, \\ \underline{c}^T x + \big(-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big) y \le b_\Delta^T \beta, \\ -\overline{c}^T x + \big(b_c^T - b_\Delta^T \operatorname{diag}(\alpha)\big) y \le b_\Delta^T \beta, \end{aligned}$$

where

$$\alpha_i := \begin{cases} \frac{|\overline{y}_i| - |\underline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \ \underline{y}_i < \overline{y}_i, \\ \operatorname{sgn}(\overline{y}_i) & \text{if } \ \underline{y}_i = \overline{y}_i, \end{cases}$$

$$\beta_i := \begin{cases} \frac{\overline{y}_i |\underline{y}_i| - \underline{y}_i |\overline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \ \underline{y}_i < \overline{y}_i, \\ 0 & \text{if } \ \underline{y}_i = \overline{y}_i. \end{cases}$$

# Contractor

## Algorithm (Optimal solution set contractor)

1. Compute an initial interval enclosure $\mathbf{x}^0, \mathbf{y}^0$
2. $i := 0$;
3. **repeat**
   1. compute the interval hull $\mathbf{x}^i, \mathbf{y}^i$ of the linearized system;
   2. $i := i + 1$;
4. **until** improvement is nonsignificant;
5. **return** $\mathbf{x}^i$;

## Properties

- Each iteration requires compting the interval hull ($2(m + n)$ linear programs).
- In practice, it converges quickly, but not to $\mathcal{S}$ in general.

# Example

## Example

Consider an interval linear program

$$
\begin{aligned}
\min -[15,16]x_1 - [17,18]x_2 \ \ \text{subject to}& \\
x_1 &\leq [10,11], \\
-x_1 + [5,6]x_2 &\leq [25,26], \\
[6,6.5]x_1 + [3,4.5]x_2 &\leq [81,82], \\
-x_1 &\leq -1, \\
x_1 - [10,12]x_2 &\leq -[1,2].
\end{aligned}
$$

Take the initial enclosure

$$
\begin{aligned}
\mathbf{x}^0 &= 1000 \cdot ([-1,1],\ [-1,1])^T, \\
\mathbf{y}^0 &= 1000 \cdot ([0,1],\ [0,1],\ [0,1],\ [0,1],\ [0,1])^T.
\end{aligned}
$$

# Example

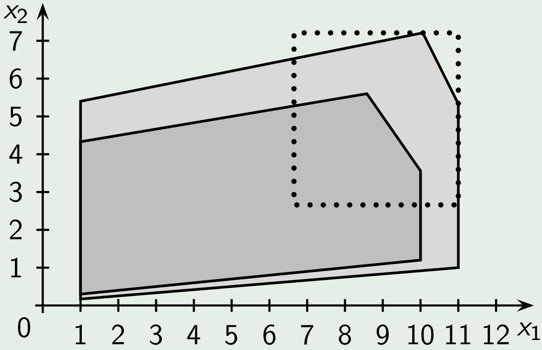
## Example (cont.)

The iterations of the procedure go as follows

$$
\begin{aligned}
\mathbf{x}^0 &= 1000 \cdot ([-1, 1], [-1, 1])^T, \\
\mathbf{y}^0 &= 1000 \cdot ([0, 1], [0, 1], [0, 1], [0, 1], [0, 1])^T, \\
\mathbf{x}^1 &= ([1, 11], [-568, 916])^T, \\
\mathbf{y}^1 &= ([0, 1000], [0, 936], [0, 358], [0, 1000], [0, 572])^T, \\
\mathbf{x}^2 &= ([1, 11], [-17.2, 72])^T, \\
\mathbf{y}^2 &= ([0, 190], [0, 58.5], [0, 24.3], [0, 176], [0, 34.6])^T, \\
\mathbf{x}^3 &= ([3.78, 11], [1.91, 9.80])^T, \\
\mathbf{y}^3 &= ([0, 30.6], [0, 6.98], [4.71], [0, 17.1], [0, 3.09])^T, \\
\mathbf{x}^4 &= ([6.65, 11], [2.66, 7.21])^T, \\
\mathbf{y}^4 &= ([0, 22.5], [0.08, 4.33], [0, 3.67], [0, 8.81], [0, 1.47])^T.
\end{aligned}
$$

# Example

## Example (cont.)




- In grey the largest and the smallest feasible area.
- The final enclosure of the optimal solution set $\mathcal{S}$ is dotted.

# Basis stability

## Definition

The interval linear programming problem

$$\min \mathbf{c}^T x \;\text{ subject to }\; \mathbf{A}x = \mathbf{b},\; x \geq 0,$$

is $B$-stable if $B$ is an optimal basis for each scenario.

## Theorem

*$B$-stability implies that the optimal value bounds are*

$$\underline{f} = \min\; \underline{c}_B^T x \;\; \textit{subject to} \;\; \underline{A}_B x_B \leq \overline{b},\; -\overline{A}_B x_B \leq -\underline{b},\; x_B \geq 0,$$
$$\overline{f} = \max\; \overline{c}_B^T x \;\; \textit{subject to} \;\; \underline{A}_B x_B \leq \overline{b},\; -\overline{A}_B x_B \leq -\underline{b},\; x_B \geq 0.$$

*Under the unique $B$-stability, the set of all optimal solutions reads*

$$\underline{A}_B x_B \leq \overline{b},\; -\overline{A}_B x_B \leq -\underline{b},\; x_B \geq 0,\; x_N = 0.$$

# Basis stability

## Non-interval case

Basis $B$ is optimal iff

C1. $A_B$ is non-singular;

C2. $A_B^{-1} b \geq 0$;

C3. $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

## Interval case

The problem is B-stable iff C1–C3 holds for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

# Basis stability

**Theorem**

*Condition C3 holds true if and only if for each $q \in \{\pm 1\}^m$ the polyhedral set described by*

$$\begin{aligned} ((A_c)_B^T - (A_\Delta)_B^T \operatorname{diag}(q))y &\le \overline{c}_B, \\ -((A_c)_B^T + (A_\Delta)_B^T \operatorname{diag}(q))y &\le -\underline{c}_B, \\ \operatorname{diag}(q)\, y &\ge 0 \end{aligned}$$

*lies inside the polyhedral set*

$$((A_c)_N^T + (A_\Delta)_N^T \operatorname{diag}(q))y \le \underline{c}_N, \ \operatorname{diag}(q)\, y \ge 0.$$

# Conclusion

## Open problems

- A sufficient and necessary condition for weak unboundedness, strong boundedness and weak optimality.
- A method to check if a given $x^* \in \mathbb{R}^n$ is an optimal solution for some scenario.
- A method for determining the image of the optimal value function.
- A sufficient and necessary condition for duality gap to be zero for each scenario.
- A method to test if a basis $B$ is optimal for some scenario.
- Tight enclosure to the optimal solution set.