On necessarily efficient solutions in interval multiobjective linear programming

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Introduction

A multiobjective linear programming problem

Consider

$$\max_{x \in \mathcal{M}} \; Cx,$$

where $\mathcal{M} = \{x \in \mathbb{R}^n \mid A \leq b\}$, $C \in \mathbb{R}^{s \times n}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

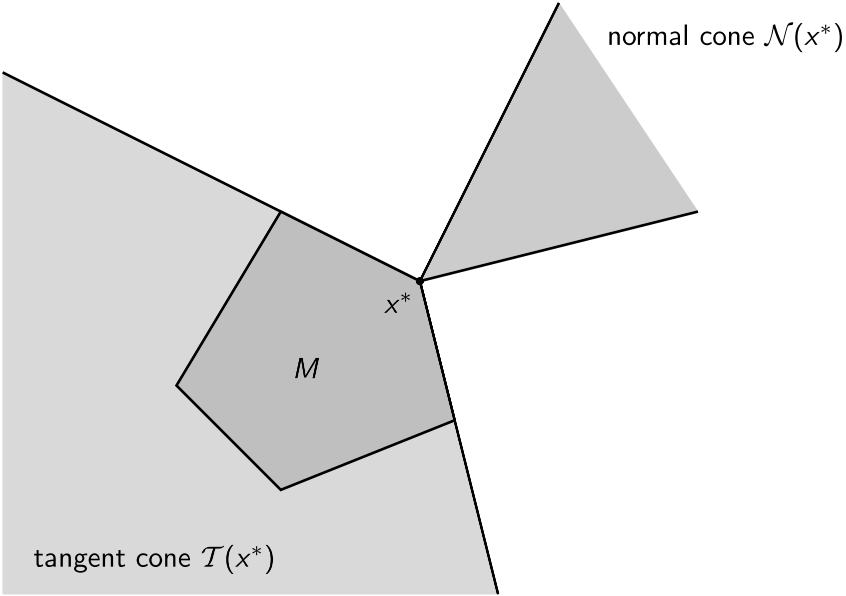
An interval problem

Suppose

$$C \in \mathbf{C} = [\underline{C}, \overline{C}] = [C_c - C_\Delta, C_c + C_\Delta].$$

$x^* \in \mathcal{M}$ is necessarily efficient if it is efficient for every $C \in \mathbf{C}$.

Introduction



Introduction

Tangent cone

The tangent cone of $\mathcal{M}$ at the point x^* is defined

$$\mathcal{T}(x^*) := \{x \in \mathbb{R}^n \mid A_P x \leq 0\},$$

where $P := \{i \mid A_{i*}x^* = b_i\}$.

Normal cone

The normall cone of $\mathcal{M}$ at the point x^* is defined

$$\mathcal{N}(x^*) : = \{x \in \mathbb{R}^n \mid x^T y \leq 0 \ \forall y \in \mathcal{T}(x^*)\} = \{x \in \mathbb{R}^n \mid Dx \geq 0\}$$

for some $D \in \mathbb{R}^{r \times n}$.

Remark

If x^* is a non-degenerate basic solution corresponding to a basis $B \subseteq \{1, \dots, m\}$ then $D = -(A_B^T)^{-1}$.

Necessarily efficiency

Theorem

The vector x^* is necessarily efficient iff the system

$$C^c x + C^\Delta |x| \gneqq 0,\ \ A_P x \le 0,\ \ e^T |x| = 1$$

has no solution.

...gives rise to an exponential algorithm

Sufficient condition

Theorem (sufficient condition)

Define $M := \overline{D\mathbf{C}^T}$. *If the linear system*

$$M\lambda \leq 0, \ \ \lambda \geq e$$

is solvable then x^* *is necessarily efficient.*

Remark

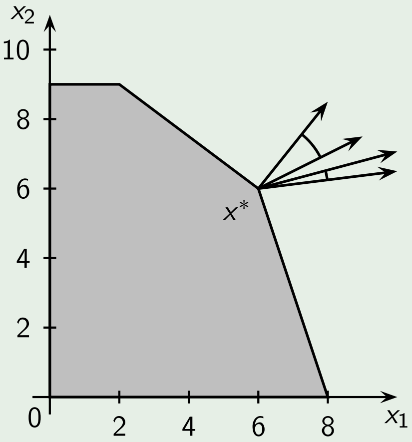
$$m_{ij} := \sum_{k=1}^{n} d_{ik} c_{kj}(d_{ik}), \quad \text{where} \quad c_{kj}(d_{ik}) := \begin{cases} \overline{c}_{kj} & \text{if } d_{ik} \geq 0, \\ \underline{c}_{kj} & \text{if } d_{ik} < 0. \end{cases}$$

Remark

For x^* a non-degenerate basic solution we get the Bitran's condition.

Sufficient condition

Example (Inuiguchi and Sakawa, 1996)



Sufficient condition

Example (Inuiguchi and Sakawa, 1996)

Let

$$\mathbf{C} = \begin{pmatrix} [2,3] & [1.5,2.5] \\ [3,4] & [0.5,0.8] \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 4 \\ 3 & 1 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 42 \\ 24 \\ 9 \\ 0 \\ 0 \end{pmatrix},$$

and a non-degenerate $x^* = (6,6)^T$ corresponding to the basis $B = \{1,2\}$. The normal cone at x^*:

$$\mathcal{N}(x^*) = \{x \mid -(A_B^T)^{-1} x \leq 0\} = \{x \mid x_1 - 3x_2 \leq 0,\ -4x_1 + 3x_2 \leq 0\}.$$

Now, the linear system

$$-1.5\lambda_1 + 2.5\lambda_2 \leq 0,\ -0.5\lambda_1 - 9.6\lambda_2 \leq 0,\ \lambda_1, \lambda_2 \geq 1$$

is solvable (e.g. $\lambda = (2,1)^T$). Thus $(6,6)^T$ is necessarily efficient.

Necessary condition

Theorem (necessary condition)

Let $k \in P$. Put $d := A_{k,*} - \sum_{i \in P,\, i \neq k} A_{i,*}$ and define $C^0 \in \mathbf{C}$ column-wise as follows

$$C^0_{*,j} = \begin{cases} \overline{C}_{*,j} & \text{if } d_j \geq 0, \\ \underline{C}_{*,j} & \text{otherwise}, \end{cases}$$

$j = 1, \ldots, n$. If the linear system

$$C^0 x - y \geq 0, \;\; A_P x \leq 0, \;\; y \geq 0, \;\; e^T y = 1.$$

is solvable then x^* is not necessarily efficient.

Remark

- It requires to solve at most m linear programs.
- Another necessary condition: for no feasible x^0 one has

$$C_c(x^0 - x^*) + C_\Delta |x^0 - x^*| \gneqq 0.$$

Necessary condition

Example (Ida, 1999, Oliveira and Antunes, 2007)

Let

$$\max\ \mathbf{C}x \ \text{ subject to } \ Ax \le b,\ x \ge 0$$

where

$$\mathbf{C} = \begin{pmatrix} [1,2] & [2,3] & [-2,-1] & [3,4] & [2,3] & [0,1] & [1,2] \\ [-1,0] & [1,2] & [1,2] & [2,3] & [3,4] & [1,2] & [0,1] \\ [3,4] & [0,1] & [1,2] & [1,2] & [0,1] & [-2,-1] & [-2,-1] \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & -1 & 1 & -2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 16 \\ 16 \\ 16 \\ 16 \end{pmatrix}.$$

Consider the point $x^* = \left(0, 0, \frac{32}{3}, \frac{16}{3}, 0, 0, 0\right)^T$.

Then $P = \{1, 4, 5, 6, 9, 10, 11\}$. For $k := 1$

$$d := A_{1,*} - \sum_{i \in P,\, i \neq 1} A_{i,*} = (2, 2, -1, 2, 2, 4, 4).$$

Necessary condition

Example (cont.)

The corresponding objective function matrix reads

$$C^0 = \begin{pmatrix} 2 & 3 & -2 & 4 & 3 & 1 & 2 \\ 0 & 2 & 1 & 3 & 4 & 2 & 1 \\ 4 & 1 & 1 & 2 & 1 & -1 & -1 \end{pmatrix}$$

and the linear system has a solution $x = (0.042, 0, -0.209, 0.167, 0, 0, 0)^T$ and $y = (0.417, 0.292, 0.292)^T$. Therefore x^* is not necessarily efficient.

- For $k \in \{5, 6, 9, 10, 11\}$ we obtain the same result.
- Another way:

$$C_c(x^0 - x^*) + C_\Delta|x^0 - x^*| = (64, \tfrac{64}{3}, \tfrac{32}{3})^T \gneqq 0.$$

  for the neighboring vertex $x^0 = (0, 0, 0, 16, 0, 0, 0)^T$.

Numerical experiments

- In accordance with Bitran (1980) we considered

$$\max_{x \in \mathcal{M}} \; Cx, \qquad C \in \mathbf{C},$$

where

$$\mathcal{M} := \{x \in \mathbb{R}^n \mid Ax = b,\, x \geq 0\}.$$

  - Entries of $A \in \mathbb{R}^{n \times 2n}$ uniformly from $[-10, 10]$, the first row from $[0, 20]$,
  - $b := Ae$,
  - entries of $C_c \in \mathbb{R}^{s \times 2n}$ uniformly from $[-10, 10]$,
  - entries of $C_\Delta \in \mathbb{R}^{s \times 2n}$ in $[0, R]$, where $R > 0$ was a parameter.
- Testing $(s+1)$ solutions to

$$\max_{x \in \mathcal{M}} e^T C_c x$$

and

$$\max_{x \in \mathcal{M}} e_k^T C_c x, \;\; k = 1, \ldots, s.$$

- In each setting of s, n and R we carried out a sequence of 50 runs.

Numerical experiments

n	s	R	sufficient cond.	efficient	necessary cond.	non-efficient
10	2	0.01	139	140	6	10
10	4	0.01	242	242	4	8
10	6	0.01	345	345	3	5
10	2	0.1	18	26	75	124
10	4	0.1	60	109	55	141
10	6	0.1	103	217	39	133
10	2	1	0	0	150	150
10	4	1	0	0	250	250
10	6	1	0	0	350	350
15	2	0.01	112	115	12	35
15	4	0.01	230	237	6	13
15	6	0.01	336	342	2	8
15	2	0.1	4	8	116	142
15	4	0.1	4	27	121	223
15	6	0.1	17	101	85	249
15	2	1	0	0	150	150
15	4	1	0	0	250	250
15	6	1	0	0	350	350

Conclusion and future work

Conclusion

- The sufficient condition is effective, particularly for thin intervals.
- The necessary condition is effective, particularly for wide intervals.

$\Rightarrow$ acceleration in testing necessarily efficiency.

Future work

- Intervals in the constraints.