A contractor for the symmetric solution set

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Interval linear systems

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\},$$

and the corresponding center and radius matrices

$$A^c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Interval linear systems

Abbreviated by $\mathbf{A}x = \mathbf{b}$, and meaning a family

$$Ax = b, \quad A \in \mathbf{A},\ b \in \mathbf{b}.$$

The solution set is

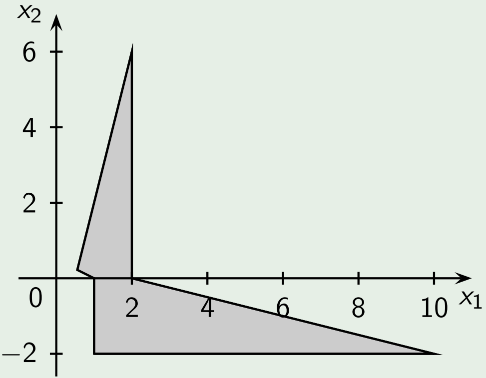
$$\Sigma = \{x \in \mathbb{R}^n \mid Ax = b,\ A \in \mathbf{A},\ b \in \mathbf{b}\}.$$

Interval linear systems

Example

Consider

$$\begin{pmatrix} [1,2] & [0,4] \\ [0,4] & -1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



Interval linear systems

Goal

Find a tight enclosure to the solution set Σ (i.e. an interval vector containing Σ).

Theorem (Oettli and Prager, 1964)

The solution set Σ is described by

$$|A^c x - b^c| \leq A^\Delta |x| + b^\Delta.$$

Theorem (Rohn & Kreinovich, 1995)

Finding the optimal enclosure (interval hull) is NP-hard.

Interval linear systems

Methods

Direct:

- Interval Gaussian elimination
- Hansen–Bliek–Rohn method (Hansen, 1992, Bliek, 1992, Rohn, 1993)

Iterative:

- Interval Gauss–Seidel algorithm
- Krawczyk iteration

Symmetric interval linear systems

The symmetric solution set

$$\Sigma_{sym} = \{x \in \mathbb{R}^n \mid Ax = b,\ A \in \mathbf{A},\ A = A^T,\ b \in \mathbf{b}\}.$$

Methods

- Interval Cholesky method (Alefeld & Mayer, 1993, 2008)
- General linear dependence solvers (Jansson, 1991, Rump, 1994, Popova, 2004, 2007, Popova & Krämer, 2007, Kolev 2004, 20006)

Applications

- Truss mechanics
- Nodal analysis for linear electrical circuits
- Eigenvalue problems

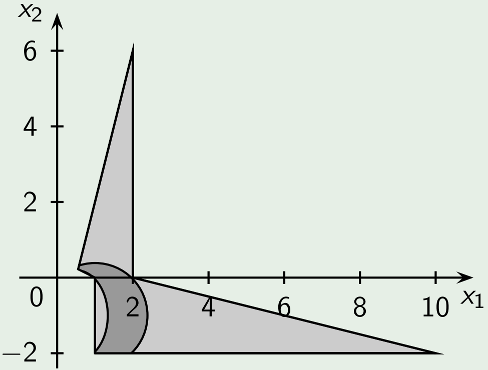
Symmetric interval linear systems: An example

Example

Consider

$$\begin{pmatrix} [1,2] & [0,4] \\ [0,4] & -1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Solution set Σ and the symmetric solution set Σ_{sym}:



Symmetric interval linear systems: Our approach

Algorithm (Symmetric solution set contractor)

1. Compute an initial interval enclosure $\mathbf{x}^0 \supseteq \Sigma_{sym}$;
2. $i := 0$;
3. **repeat**
    1. compute a polyhedral enclosure $\mathcal{P}$ of Σ_{sym} by using $\mathbf{x}^i$;
    2. $i := i + 1$;
    3. compute the interval hull $\mathbf{x}^i$ of $\mathcal{P}$;
4. **until** improvement is nonsignificant;
5. **return** $\mathbf{x}^i$;

Description of the symmetric solution set

Theorem (Hladík, 2008)

The symmetric solution set Σ_{sym} is described by the following system of inequalities

$$A^\Delta|x| + b^\Delta \geq |b^c - A^c x|,$$

$$\sum_{i,j=1}^{n} a_{ij}^\Delta |x_i x_j (p_i - q_j)| + \sum_{i=1}^{n} b_i^\Delta |x_i(p_i + q_i)| \geq \left| \sum_{i=1}^{n} (b_i^c - A_{i,*}^c x) x_i (p_i - q_i) \right|$$

for all vectors $p, q \in \{0,1\}^n \setminus \{0,1\}$ such that

$$p \prec_{\text{lex}} q \text{ and } (p = 1 - q \vee \exists i : p_i = q_i = 0).$$

Linearization of bilinear forms

Theorem (Adjiman et al., 1998)

For every $x \in \mathbf{x} \subset \mathbb{R}$ *and* $y \in \mathbf{y} \subset \mathbb{R}$ *we have*

$$
\begin{aligned}
xy &\le \overline{x}y + \underline{y}x - \overline{x}\underline{y},\\
xy &\le \underline{x}y + \overline{y}x - \underline{x}\overline{y},\\
xy &\ge \underline{x}y + \underline{y}x - \underline{x}\underline{y},\\
xy &\ge \overline{x}y + \overline{y}x - \overline{x}\overline{y}.
\end{aligned}
$$

Linearization of absolute values

Theorem (Beaumont, 1998)

For every $x \in \mathbf{x} \subset \mathbb{R}$ *with* $\underline{x} < \overline{x}$ *we have*

$$|x| \leq \alpha x + \beta, \tag{1}$$

where

$$\alpha = \frac{|\overline{x}| - |\underline{x}|}{\overline{x} - \underline{x}} \quad \textit{and} \quad \beta = \frac{\overline{x}|\underline{x}| - \underline{x}|\overline{x}|}{\overline{x} - \underline{x}}.$$

Moreover, if $\underline{x} \geq 0$ *or* $\overline{x} \leq 0$ *then* (1) *holds as equation.*

Selection of inequalities

Selection of p, q

(S1) $p = e_k$ and $q = e_l$, where $k = 1, \dots, n$, $l = k+1, \dots, n$,

(S2) $p = e_k$ and $q = 1 - p$, where $k = 1, \dots, n$,

(S3) make $\frac{1}{4}n^2 + 2n$ random selections of $p, q \in \{0,1\}^n$ with probabilities:

$$P(p_i = 0) = \frac{4}{7},\ P(p_i = 1) = \frac{3}{7},$$
$$P(q_i = 0) = P(q_i = 1) = \frac{1}{2},$$

(S4) $2n$ more selections to cut off possibly large part of polyhedron.

Summary

We use $3n^2 + 20n$ inequalities in total.

Numerical examples

Example (Behnke, 1989)

Consider

$$\mathbf{A} = \begin{pmatrix} 3 & [1,2] \\ [1,2] & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} [10, 10.5] \\ [10, 10.5] \end{pmatrix}.$$

Results and comparisons:

our algorithm	$([1.740, 2.726], [1.740, 2.726])^T$
The interval hull of Σ_{sym}	$([1.800, 2.687], [1.810, 2.688])^T$
The interval hull of Σ	$([1.285, 3.072], [1.285, 3.072])^T$
The Rump inclusion method	$([1.623, 2.932], [1.623, 2.932])^T$
The interval Cholesky method	$([0.467, 3.125], [1.125, 4.300])^T$

Numerical examples

Example (Random symmetric systems)

Data were generated randomly as follows:

- A^c_{ij} chosen randomly and independently in $[-10, 10]$,
- $A^\Delta_{ij} := R$ randomly in $[0, R]$, where $R > 0$ is a parameter,
- symmetrize $A^c := A^c + (A^c)^T + 20nI$, $A^\Delta := A^\Delta + (A^\Delta)^T$,
- b^c_i randomly in $[-10n, 10n]$, b^Δ_i in $[0, R]$.

Note

- Computations in MATLAB with INTLAB,
  - `verifylss` computes a fast interval enclosure of Σ,
  - `verintervalhull` computes the verified interval hull of Σ.
- Efficiency measured by the volume that is cut off, i.e.

$$\frac{\prod_{i=1}^n (x_i^0)^\Delta - \prod_{i=1}^n (x_i^*)^\Delta}{\prod_{i=1}^n (x_i^0)^\Delta} 100\%.$$

Numerical examples

Example (cont'd)

n	R	runs	exec. time	`verifylss` cut off	`verintervalhull` cut off
5	0.1	100	5.13 s	21.8 %	18.8 %
5	0.5	100	5.52 s	29.5 %	15.6 %
5	1	100	5.71 s	38.0 %	13.1 %
10	0.1	100	56.3 s	36.1 %	31.3 %
10	0.5	100	54.5 s	47.5 %	25.5 %
10	1	100	55.4 s	57.8 %	19.4 %
15	0.1	100	218 s	43.6 %	37.1 %
15	0.5	100	222 s	59.6 %	31.5 %
15	1	100	211 s	72.2 %	23.9 %
20	0.1	50	604 s	51.7 %	44.1 %
20	0.5	50	600 s	68.3 %	36.3 %
20	1	50	573 s	80.9 %	26.5 %
25	0.1	50	1318 s	57.7 %	49.3 %
25	0.5	50	1312 s	75.3 %	41.0 %
25	1	50	1250 s	86.9 %	30.8 %