Interval valued bimatrix games

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Bimatrix game

- Bimatrix game is \((A, B)\) with positive matrices \(A, B \in \mathbb{R}^{m \times n}\);
- Mixed strategy for player I: \(x \in \mathbb{R}^m, x \geq 0, e^T x = 1\);
- Mixed strategy for player II: \(y \in \mathbb{R}^n, y \geq 0, e^T y = 1\);
- Expected reward for player I: \(x^T Ay\);
- Expected reward for player II: \(x^T By\);
- \((\hat{x}, \hat{y})\) is a \((Nash)\) equilibrium if
  \[
  \hat{x}^T A\hat{y} \geq x^T A\hat{y},
  \hat{x}^T B\hat{y} \geq \hat{x}^T By
  \]
  for any mixed strategy \(x\) and \(y\);
- Every bimatrix game has an equilibrium.
Theorem (Audet et al., 2006)

Let

\[
L_1 := \max_{i,j} a_{ij} - \min_{i,j} a_{ij},
\]

\[
L_2 := \max_{i,j} b_{ij} - \min_{i,j} b_{ij}.
\]

The set of equilibria is the set of mixed strategies \((x, y)\) for which there are \(\alpha, \beta \in \mathbb{R}\) and vectors \(u \in \{0, 1\}^m\) and \(v \in \{0, 1\}^n\) satisfying

\[
e^T x = 1, \quad x \geq 0,
\]

\[
e^T y = 1, \quad y \geq 0,
\]

\[
\alpha e - L_1 u \leq A y \leq \alpha e,
\]

\[
\beta e - L_2 v \leq B^T x \leq \beta e,
\]

\[
x + u \leq e, \quad y + v \leq e.
\]
Introduction

Definition

- An interval matrix

\[ A := [A, \bar{A}] = \{ A \in \mathbb{R}^{n \times n} \mid A \leq A \leq \bar{A} \}; \]

- An interval bimatrix game is \((A, B)\);

- An instance of \((A, B)\) is any \((A, B)\) with \(A \in A\) and \(B \in B\).

Example

\[ A = \begin{pmatrix} 5 & 0 \\ [4, 6] & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & [4, 6] \\ 0 & 1 \end{pmatrix}. \]

- \((A, B)\) has three equilibria \((e^1, e^1)\), \((e^2, e^2)\) and \(((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))\) with rewards respectively 5, 1 and \(\frac{5}{2}\) for both players.

- \((\bar{A}, \bar{B})\) has one equilibrium \((e^2, e^2)\) and both players earn 1.
Strong equilibria

Definition

Strong equilibrium is an equilibrium common for all instances.

Theorem (Strong equilibrium in pure strategies)

There exists a strong equilibrium in pure strategies if and only if there is some \( i \in \{1,\ldots,m\} \) and \( j \in \{1,\ldots,n\} \) such that

\[
\begin{align*}
    a_{ij} &\geq a_{kj} & \forall k = 1,\ldots,m, \ k \neq i, \\
    b_{ij} &\geq b_{ik} & \forall k = 1,\ldots,n, \ k \neq j.
\end{align*}
\]

In this case, \((e^i, e^j)\) is a strong equilibrium.
Definition

\[ L_1 := \max_{i,j} \bar{a}_{ij} - \min_{i,j} a_{ij}, \]
\[ L_2 := \max_{i,j} \bar{b}_{ij} - \min_{i,j} b_{ij}. \]

Theorem (Strong equilibrium in non-pure strategies)

A pair of mixed non-pure strategies \((\hat{x}, \hat{y})\) is a strong equilibrium iff there are \(\hat{\alpha}, \hat{\beta} \in \mathbb{R}, \hat{u} \in \{0,1\}^m\) and \(\hat{v} \in \{0,1\}^n\) solving

\[ e^T x = 1, \quad x \geq 0, \]
\[ e^T y = 1, \quad y \geq 0, \]
\[ \alpha e - L_1 u \leq Ay, \quad \bar{A} y \leq \alpha e, \]
\[ \beta e - L_2 v \leq B^T x, \quad \bar{B}^T x \leq \beta e, \]
\[ x + u \leq e, \quad y + v \leq e. \]
Strong equilibria

Theorem (Strong equilibrium in pure and non-pure strategy)

A pair $(\hat{x}, \hat{y})$ is a strong equilibrium consisting of pure strategy $\hat{x}$ and a non-pure strategy $\hat{y}$ iff there are $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0, 1\}^m$ and $\hat{v} \in \{0, 1\}^n$ solving the mixed integer linear system

$$
e^T x = 1, \quad x \geq 0,$$
$$e^T y = 1, \quad y \geq 0,$$
$$\alpha e - L_1 u \leq A y, \quad A y \leq \alpha e + L_1 (e - u),$$
$$e^T u = m - 1,$$
$$\beta e - L_2 v \leq B^T x, \quad B^T x \leq \beta e,$$
$$x + u \leq e,$$
$$y + v \leq e.$$
A pair \((\hat{x}, \hat{y})\) is a strong equilibrium iff there are \(\hat{\alpha}, \hat{\beta} \in \mathbb{R}, \hat{\gamma}, \hat{\delta} \in \{0, 1\}\), \(\hat{u} \in \{0, 1\}^m\) and \(\hat{v} \in \{0, 1\}^n\) solving

\[
\begin{align*}
e^T x &= 1, \quad x \geq 0, \\
e^T y &= 1, \quad y \geq 0, \\
\alpha e - L_1 u &\leq \bar{A}y, \quad \bar{A}y \leq \alpha e + L_1 (e - u), \\
\bar{A}y &\leq \alpha e + L_1 \gamma e, \\
(m - 1) \gamma &\leq e^T u, \\
\beta e - L_2 v &\leq \bar{B}^T x, \quad \bar{B}^T x \leq \beta e + L_2 (e - v), \\
\bar{B}^T x &\leq \beta e + L_2 \delta e, \\
(n - 1) \delta &\leq e^T v, \\
x + u &\leq e, \quad y + v \leq e.
\end{align*}
\]
Strong equilibria

Example

Consider an interval bimatrix game \((A, B)\) with

\[
A = \begin{pmatrix}
42 & [21, 24] & 21 \\
7 & [77, 80] & 35
\end{pmatrix}, \quad B = A^T.
\]

- In pure strategies, there is a unique strong equilibrium \((x, y)\) with \(x = y = (0, 0, 1)^T\). Players’ rewards are 35.
  The corresponding solution to the system consists of \(x, y, u = v = (1, 1, 0)^T, \gamma = \delta = 1\) and any \(\alpha, \beta \in [21, 35]\).

- In non-pure strategies, there is a unique strong equilibrium \((x, y)\) with \(x = y = (0.2857, 0, 0.7143)^T\) and players’ rewards \(\alpha = \beta = 27\).
  That is, \(x, y, \alpha, \beta, u = v = (0, 1, 0)^T\) and \(\gamma = \delta = 0\) form a solution the system.
Equilibria set

**Theorem**

The set of all equilibria for all bimatrix games \((A, B)\) with \(A \in \mathbf{A}\) and \(B \in \mathbf{B}\) is described by the mixed integer linear system

\[
\begin{align*}
e^T x &= 1, \quad x \geq 0, \\
e^T y &= 1, \quad y \geq 0, \\
\alpha e - L_1 u &\leq \bar{A} y, \quad A y \leq \alpha e, \\
\beta e - L_2 v &\leq \bar{B}^T x, \quad B^T x \leq \beta e, \\
x + u &\leq e, \quad u \in \{0, 1\}^m, \\
y + v &\leq e, \quad v \in \{0, 1\}^n.
\end{align*}
\]

**Consequences**

- Checking if \((x, y)\) is an equilibrium is easy;
- The equilibria set forms a union of finitely many convex polyhedra;
- Computable the range of possible rewards.
 Recall an interval bimatrix game \((A, B)\) with

\[
A = \begin{pmatrix}
42 & [21, 24] & 21 \\
7 & [77, 80] & 35
\end{pmatrix}, \quad B = A^T.
\]

Let \(u = v = (0, 0, 0)^T\). The polyhedron \(\mathcal{X}\) corresponding to variables \(x\) and \(\beta\) has vertices

\[
(x^1, \beta^1) = (0.2857, 0, 0.7143, 27.0000),
\]
\[
(x^2, \beta^2) = (0.3714, 0.1000, 0.5286, 29.1000),
\]
\[
(x^3, \beta^3) = (0.3671, 0.0886, 0.5443, 28.7089),
\]
\[
(x^4, \beta^4) = (0.3676, 0.1029, 0.5294, 29.0294),
\]
\[
(x^5, \beta^5) = (0.3636, 0.0909, 0.5455, 28.6364).
\]

The polyhedron \(\mathcal{Y}\) corresponding to \(y\) and \(\alpha\) equals \(\mathcal{X}\).
For $u = (0, 0, 0)^T$ and $v = (0, 1, 0)^T$ we calculate the set of equilibria $\mathcal{X} \times \{(0.2857, 0, 0.7143, 27.0000)\}$. It is a subset of $\mathcal{X} \times \mathcal{X}$.

Situation $u = (0, 1, 0)^T$ and $v = (0, 0, 0)^T$ is symmetric to the previous one.

For $u = (0, 1, 0)^T$ and $v = (0, 1, 0)^T$ we obtain the set of equilibria

$$\{(0.2857, 0, 0.7143, 27.0000)\} \times \{(0.2857, 0, 0.7143, 27.0000)\}.$$

Also this case is covered by the first one.

For $u = (1, 1, 0)^T$ and $v = (1, 1, 0)^T$ we get only one equilibrium $(e_3, e_3)$. The reward is 35 for both players.

The equilibria set is $(\mathcal{X} \times \mathcal{X}) \cup \{(0, 0, 1, 35, 0, 0, 1, 35)\}$.