Interval valued bimatrix games

Milan Hladík

Department of Applied Mathematics,
Faculty of Mathematics and Physics,
Charles University, Prague

MMEI 2009, České Budějovice
June 15–18

Introduction

Bimatrix game

- Bimatrix game is (A, B) with positive matrices $A, B \in \mathbb{R}^{m \times n}$;
- Mixed strategy for player I: $x \in \mathbb{R}^m$, $x \geq 0$, $e^T x = 1$;
- Mixed strategy for player II: $y \in \mathbb{R}^n$, $y \geq 0$, $e^T y = 1$;
- Expected reward for player I: $x^T A y$;
- Expected reward for player II: $x^T B y$;
- $(\hat{x}, \hat{y})$ is a *(Nash) equilibrium* if

$$\hat{x}^T A \hat{y} \geq x^T A \hat{y},$$
$$\hat{x}^T B \hat{y} \geq \hat{x}^T B y$$

  for any mixed strategy x and y;
- Every bimatrix game has an equilibrium.

Introduction

Theorem (Audet et al., 2006)

Let

$$L_1 := \max_{i,j} a_{ij} - \min_{i,j} a_{ij},$$
$$L_2 := \max_{i,j} b_{ij} - \min_{i,j} b_{ij}.$$

The set of equilibria is the set of mixed strategies (x, y) *for which there are* $\alpha, \beta \in \mathbb{R}$ *and vectors* $u \in \{0,1\}^m$ *and* $v \in \{0,1\}^n$ *satisfying*

$$e^T x = 1, \quad x \geq 0,$$
$$e^T y = 1, \quad y \geq 0,$$
$$\alpha e - L_1 u \leq Ay \leq \alpha e,$$
$$\beta e - L_2 v \leq B^T x \leq \beta e,$$
$$x + u \leq e, \quad y + v \leq e.$$

Introduction

Definition

- An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n} \mid \underline{A} \le A \le \overline{A}\};$$

- An interval bimatrix game is $(\mathbf{A}, \mathbf{B})$;
- An instance of $(\mathbf{A}, \mathbf{B})$ is any (A, B) with $A \in \mathbf{A}$ and $B \in \mathbf{B}$.

Example

$$\mathbf{A} = \begin{pmatrix} 5 & 0 \\ [4,6] & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 5 & [4,6] \\ 0 & 1 \end{pmatrix}.$$

- $(\underline{A}, \underline{B})$ has three equilibria (e^1, e^1), (e^2, e^2) and $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ with rewards respectively 5, 1 and $\frac{5}{2}$ for both players.
- $(\overline{A}, \overline{B})$ has one equilibrium (e^2, e^2) and both players earn 1.

Strong equilibria

Definition

Strong equilibrium is an equilibrium common for all instances.

Theorem (Strong equilibrium in pure strategies)

There exists a strong equilibrium in pure strategies if and only if there is some $i \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, n\}$ *such that*

$$\underline{a}_{ij} \geq \overline{a}_{kj} \quad \forall k = 1, \dots, m,\ k \neq i,$$
$$\underline{b}_{ij} \geq \overline{b}_{ik} \quad \forall k = 1, \dots, n,\ k \neq j.$$

In this case, (e^i, e^j) *is a strong equilibrium.*

Strong equilibria

Definition

$$L_1 := \max_{i,j} \overline{a}_{ij} - \min_{i,j} \underline{a}_{ij},$$

$$L_2 := \max_{i,j} \overline{b}_{ij} - \min_{i,j} \underline{b}_{ij}.$$

Theorem (Strong equilibrium in non-pure strategies)

A pair of mixed non-pure strategies $(\hat{x}, \hat{y})$ is a strong equilibrium iff there are $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ solving

$$e^T x = 1, \quad x \geq 0,$$

$$e^T y = 1, \quad y \geq 0,$$

$$\alpha e - L_1 u \leq \underline{A} y, \quad \overline{A} y \leq \alpha e,$$

$$\beta e - L_2 v \leq \underline{B}^T x, \quad \overline{B}^T x \leq \beta e,$$

$$x + u \leq e, \quad y + v \leq e.$$

Strong equilibria

Theorem (Strong equilibrium in pure and non-pure strategy)

A pair $(\hat{x}, \hat{y})$ is a strong equilibrium consisting of pure strategy $\hat{x}$ and a non-pure strategy $\hat{y}$ iff there are $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{u} \in \{0,1\}^m$ and $\hat{v} \in \{0,1\}^n$ solving the mixed integer linear system

$$
\begin{aligned}
e^T x &= 1, \quad x \geq 0,\\
e^T y &= 1, \quad y \geq 0,\\
\alpha e - L_1 u \leq \underline{A} y, &\quad \overline{A} y \leq \alpha e + L_1(e-u),\\
e^T u &= m-1,\\
\beta e - L_2 v \leq \underline{B}^T x, &\quad \overline{B}^T x \leq \beta e,\\
x + u &\leq e,\\
y + v &\leq e.
\end{aligned}
$$

Strong equilibria

Theorem (Strong equilibrium)

A pair $(\hat{x}, \hat{y})$ *is a strong equilibrium iff there are* $\hat{\alpha}, \hat{\beta} \in \mathbb{R}$, $\hat{\gamma}, \hat{\delta} \in \{0, 1\}$, $\hat{u} \in \{0, 1\}^m$ *and* $\hat{v} \in \{0, 1\}^n$ *solving*

$$
\begin{aligned}
e^T x = 1, &\quad x \geq 0,\\
e^T y = 1, &\quad y \geq 0,\\
\alpha e - L_1 u \leq \underline{A} y, &\quad \overline{A} y \leq \alpha e + L_1(e - u),\\
\overline{A} y &\leq \alpha e + L_1 \gamma e,\\
(m-1)\gamma &\leq e^T u,\\
\beta e - L_2 v \leq \underline{B}^T x, &\quad \overline{B}^T x \leq \beta e + L_2(e - v),\\
\overline{B}^T x &\leq \beta e + L_2 \delta e,\\
(n-1)\delta &\leq e^T v,\\
x + u \leq e, &\quad y + v \leq e.
\end{aligned}
$$

Strong equilibria

Example

Consider an interval bimatrix game $(\mathbf{A}, \mathbf{B})$ with

$$\mathbf{A} = \begin{pmatrix} 42 & [21, 24] & 21 \\ [49, 52] & [35, 38] & [14, 17] \\ 7 & [77, 80] & 35 \end{pmatrix}, \quad \mathbf{B} = \mathbf{A}^T.$$

- In pure strategies, there is a unique strong equilibrium (x, y) with $x = y = (0, 0, 1)^T$. Players' rewards are 35.
  The corresponding solution to the system consists of x, y, $u = v = (1, 1, 0)^T$, $\gamma = \delta = 1$ and any $\alpha, \beta \in [21, 35]$.
- In non-pure strategies, there is a unique strong equilibrium (x, y) with $x = y = (0.2857, 0, 0.7143)^T$ and players' rewards $\alpha = \beta = 27$.
  That is, x, y, α, β, $u = v = (0, 1, 0)^T$ and $\gamma = \delta = 0$ form a solution the system.

Equilibria set

Theorem

The set of all equilibria for all bimatrix games (A, B) *with* $A \in \mathbf{A}$ *and* $B \in \mathbf{B}$ *is described by the mixed integer linear system*

$$
\begin{gathered}
e^T x = 1,\ x \geq 0, \\
e^T y = 1,\ y \geq 0, \\
\alpha e - L_1 u \leq \overline{A} y,\ \underline{A} y \leq \alpha e, \\
\beta e - L_2 v \leq \overline{B}^T x,\ \underline{B}^T x \leq \beta e, \\
x + u \leq e, \quad u \in \{0, 1\}^m, \\
y + v \leq e, \quad v \in \{0, 1\}^n.
\end{gathered}
$$

Consequences

- Checking if (x, y) is an equilibrium is easy;
- The equilibria set forms a union of finitely many convex polyhedra;
- Computable the range of possible rewards.

Equilibria set

Example

Recall an interval bimatrix game $(\mathbf{A}, \mathbf{B})$ with

$$\mathbf{A} = \begin{pmatrix} 42 & [21, 24] & 21 \\ [49, 52] & [35, 38] & [14, 17] \\ 7 & [77, 80] & 35 \end{pmatrix}, \quad \mathbf{B} = \mathbf{A}^T.$$

1. Let $u = v = (0, 0, 0)^T$. The polyhedron $\mathcal{X}$ corresponding to variables x and β has vertices

$$(x^1, \beta^1) = (0.2857, 0, 0.7143, 27.0000),$$
$$(x^2, \beta^2) = (0.3714, 0.1000, 0.5286, 29.1000),$$
$$(x^3, \beta^3) = (0.3671, 0.0886, 0.5443, 28.7089),$$
$$(x^4, \beta^4) = (0.3676, 0.1029, 0.5294, 29.0294),$$
$$(x^5, \beta^5) = (0.3636, 0.0909, 0.5455, 28.6364).$$

The polyhedron $\mathcal{Y}$ corresponding to y and α equals $\mathcal{X}$.

Equilibria set

Example (cont.)

2. For $u = (0,0,0)^T$ and $v = (0,1,0)^T$ we calculate the set of equilibria $\mathcal{X} \times \{(0.2857, 0, 0.7143, 27.0000)\}$. It is a subset of $\mathcal{X} \times \mathcal{X}$.
3. Situation $u = (0,1,0)^T$ and $v = (0,0,0)^T$ is symmetric to the previous one.
4. For $u = (0,1,0)^T$ and $v = (0,1,0)^T$ we obtain the set of equilibria

   $$\{(0.2857, 0, 0.7143, 27.0000)\} \times \{(0.2857, 0, 0.7143, 27.0000)\}.$$

   Also this case is covered by the first one.
5. For $u = (1,1,0)^T$ and $v = (1,1,0)^T$ we get only one equilibrium (e_3, e_3). The reward is 35 for both players.

The equilibria set is $(\mathcal{X} \times \mathcal{X}) \cup \{(0, 0, 1, 35, 0, 0, 1, 35)\}$.