Support set invariancy for interval bimatrix games

Milan Hladík

Department of Applied Mathematics,
Faculty of Mathematics and Physics,
Charles University, Prague

EUROPT 2009, Remagen, Germany
July 3–4

Introduction

Bimatrix game

- Bimatrix game is (A, B) with positive matrices $A, B \in \mathbb{R}^{m\times n}$;
- Mixed strategy for player I: $x \in \mathbb{R}^m$, $x \geq 0$, $e^T x = 1$;
- Mixed strategy for player II: $y \in \mathbb{R}^n$, $y \geq 0$, $e^T y = 1$;
- Expected reward for player I: $x^T Ay$;
- Expected reward for player II: $x^T By$;
- $(\hat{x}, \hat{y})$ is a *(Nash) equilibrium* if

$$\hat{x}^T A\hat{y} \geq x^T A\hat{y},$$
$$\hat{x}^T B\hat{y} \geq \hat{x}^T By$$

  for any mixed strategy x and y;
- Every bimatrix game has an equilibrium.

Introduction

Theorem (Audet et al., 2006)

Let

$$L_1 := \max_{i,j} a_{ij} - \min_{i,j} a_{ij},$$
$$L_2 := \max_{i,j} b_{ij} - \min_{i,j} b_{ij}.$$

The set of equilibria is the set of mixed strategies (x, y) for which there are $\alpha, \beta \in \mathbb{R}$ and vectors $u \in \{0,1\}^m$ and $v \in \{0,1\}^n$ satisfying

$$e^T x = 1, \quad x \geq 0,$$
$$e^T y = 1, \quad y \geq 0,$$
$$\alpha e - L_1 u \leq Ay \leq \alpha e,$$
$$\beta e - L_2 v \leq B^T x \leq \beta e,$$
$$x + u \leq e, \quad y + v \leq e.$$

Introduction

Definition

- An interval matrix $\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n \times n} \mid \underline{A} \leq A \leq \overline{A}\}$;
- The midpoint of $\mathbf{A}$ is $A_c := \frac{1}{2}(\underline{A} + \overline{A})$;
- The radius of $\mathbf{A}$ is $A_c := \frac{1}{2}(\overline{A} - \underline{A})$;
- An interval bimatrix game is $(\mathbf{A}, \mathbf{B})$;
- An instance of $(\mathbf{A}, \mathbf{B})$ is any (A, B) with $A \in \mathbf{A}$ and $B \in \mathbf{B}$.

Example

$$\mathbf{A} = \begin{pmatrix} 5 & 0 \\ [4,6] & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 5 & [4,6] \\ 0 & 1 \end{pmatrix}.$$

- $(\underline{A}, \underline{B})$ has three equilibria (e^1, e^1), (e^2, e^2) and $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ with rewards respectively 5, 1 and $\frac{5}{2}$ for both players.
- $(\overline{A}, \overline{B})$ has one equilibrium (e^2, e^2) and both players earn 1.

Interval games

- Cooperation under interval uncertainty:

  Alparslan-Gök et al., 2008
- Zero-sum interval matrix games:

  Levin, 1999
  Shashikhin, 2004
  Collins & Hu, 2005, 2008
  Liu & Kao, 2008
- Interval bimatrix games:

  Hladík, 2009

Support set invariancy

Definition

Support of a vector x is $\sigma(x) := \{i \mid x_i \neq 0\}$.

Support set invariancies

For an interval game $(\mathbf{A}, \mathbf{B})$ and index sets $S_1 \subseteq \{1, \dots, m\}$ and $S_2 \subseteq \{1, \dots, n\}$ introduce

(SSI1) Every instance (A, B), $A \in \mathbf{A}$ and $B \in \mathbf{B}$ has an equilibrium (x, y) such that $\sigma(x) = S_1$ and $\sigma(y) = S_2$.

(SSI2) Every instance (A, B), $A \in \mathbf{A}$ and $B \in \mathbf{B}$ has an equilibrium (x, y) such that $\sigma(x) \subseteq S_1$ and $\sigma(y) \subseteq S_2$.

(SSI3) Every instance (A, B), $A \in \mathbf{A}$ and $B \in \mathbf{B}$ has an equilibrium (x, y) such that $\sigma(x) \supseteq S_1$ and $\sigma(y) \supseteq S_2$.

The first kind of support set invariancy

Theorem

Let $S_1 \subseteq \{1, \ldots, m\}$ *and* $S_2 \subseteq \{1, \ldots, n\}$. *Remove from* $\mathbf{A}$ *the columns indexed by* $\{1, \ldots, n\} \setminus S_2$ *and from* $\mathbf{B}$ *the rows indexed by* $\{1, \ldots, m\} \setminus S_1$. *Then* (SSI1) *holds true iff for every* $z^1 \in \{\pm 1\}^{|S_1|}$ *there exist* $y \in \mathbb{R}^{|S_2|}$ *and* $\alpha \in \mathbb{R}$ *satisfying*

$$
\begin{aligned}
e^T y = 1,\ y \geq \varepsilon_2 e,& \\
\alpha = (A^c_{i,\cdot} - z^1_i A^\Delta_{i,\cdot})y, &\quad \forall i \in S_1,\\
\alpha \geq (\overline{A}_{i,\cdot})y, &\quad \forall i \notin S_1,
\end{aligned}
$$

and for every $z^2 \in \{\pm 1\}^{|S_2|}$ *there exist* $x \in \mathbb{R}^{|S_1|}$ *and* $\beta \in \mathbb{R}$ *satisfying*

$$
\begin{aligned}
e^T x = 1,\ x \geq \varepsilon_1 e,& \\
\beta = (B^c_{\cdot,i} - z^2_i B^\Delta_{\cdot,i})^T x, &\quad \forall i \in S_2,\\
\beta \geq (\overline{B}_{\cdot,i})^T x, &\quad \forall i \notin S_2,
\end{aligned}
$$

where $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *are sufficiently small.*

The second kind of support set invariancy

Theorem (sufficient condition)

Let $S_1 \subseteq \{1, \dots, m\}$ *and* $S_2 \subseteq \{1, \dots, n\}$. *Remove from* $\mathbf{A}$ *the columns indexed by* $\{1, \dots, n\} \setminus S_2$ *and from* $\mathbf{B}$ *the rows indexed by* $\{1, \dots, m\} \setminus S_1$. *Then* (SSI2) *holds true if for every* $k \in \{1, \dots, m\} \setminus S_1$ *the linear system*

$$\sum_{i \in S_1} \lambda_i \underline{A}_{i,\cdot} \geq \overline{A}_{k,\cdot},$$
$$\sum_{i \in S_1} \lambda_i = 1, \quad \lambda_i \geq 0 \ \ \forall i \in S_1$$

is solvable, and for every $k \in \{1, \dots, n\} \setminus S_2$ *there is a solution to*

$$\sum_{i \in S_2} \lambda_i \underline{B}_{\cdot,i} \geq \overline{B}_{\cdot,k},$$
$$\sum_{i \in S_2} \lambda_i = 1, \quad \lambda_i \geq 0 \ \ \forall i \in S_2.$$

The second kind of support set invariancy

Theorem (sufficient condition)

Let $S_1 \subseteq \{1,\dots,m\}$ and $S_2 \subseteq \{1,\dots,n\}$. Remove from **A** *the columns indexed by $\{1,\dots,m\} \setminus S_2$ and from* **B** *the rows indexed by $\{1,\dots,n\} \setminus S_1$. Then* (SSI2) *holds true if for every $z^1 \in \{\pm 1\}^{|S_1|}$ there are $y \in \mathbb{R}^{|S_2|}$ and $\alpha \in \mathbb{R}$ satisfying the linear system*

$$\begin{aligned} & e^T y = 1,\ y \geq 0, \\ & \quad \alpha = (A^c_{i,\cdot} - z^1_i A^\Delta_{i,\cdot})y, \quad \forall i \in S_1, \\ & \quad \alpha \geq (\overline{A}_{i,\cdot})y, \quad \forall i \notin S_1, \end{aligned}$$

and for every $z^2 \in \{\pm 1\}^{|S_2|}$ there are $x \in \mathbb{R}^{|S_1|}$ and $\beta \in \mathbb{R}$ satisfying

$$\begin{aligned} & e^T x = 1,\ x \geq 0, \\ & \quad \beta = (B^c_{\cdot,i} - z^2_i B^\Delta_{\cdot,i})^T x, \quad \forall i \in S_2, \\ & \quad \beta \geq (\overline{B}_{\cdot,i})^T x, \quad \forall i \notin S_2. \end{aligned}$$

The third kind of support set invariancy

Theorem (necessary condition)

Let $S_1 \subseteq \{1, \dots, m\}$ and $S_2 \subseteq \{1, \dots, n\}$. If (SSI3) *holds true then for every $z^1 \in \{\pm 1\}^m$ there exist $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ satisfying*

$$\begin{aligned} e^T y &= 1, \\ y_i &\geq \varepsilon_2, \quad \forall i \in S_2, \\ \alpha &= (A^c_{i,\cdot} - z^1_i A^\Delta_{i,\cdot}) y, \quad \forall i \in S_1, \\ \alpha &\geq (\overline{A}_{i,\cdot}) y, \quad \forall i \notin S_1, \end{aligned}$$

for every $z^2 \in \{\pm 1\}^n$ there exist $x \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$ satisfying

$$\begin{aligned} e^T x &= 1, \\ x_i &\geq \varepsilon_1, \quad \forall i \in S_1, \\ \beta &= (B^c_{\cdot,i} - z^2_i B^\Delta_{\cdot,i})^T x, \quad \forall i \in S_2, \\ \beta &\geq (\overline{B}_{\cdot,i})^T x, \quad \forall i \notin S_2, \end{aligned}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are sufficiently small.

Support set invariancy – an example

Example

Consider an interval bimatrix game $(\mathbf{A}, \mathbf{B})$ with

$$\underline{A} := \begin{pmatrix} 0 & 25 & 5 & 5 \\ 40 & 0 & 5 & 10 \\ 10 & 15 & 20 & 0 \\ 20 & 5 & 10 & 15 \end{pmatrix}, \quad \overline{A} := \begin{pmatrix} 5 & 28 & 8 & 8 \\ 43 & 5 & 8 & 13 \\ 13 & 18 & 25 & 3 \\ 23 & 8 & 13 & 20 \end{pmatrix},$$

$$\underline{B} := \underline{A}^T, \quad \overline{B} := \overline{A}^T.$$

The bimatrix game $(\underline{A}, \underline{B})$ has equilibria

1. (x^1, y^1), $x^1 = y^2 = (\frac{1}{3}, \frac{8}{15}, \frac{2}{15}, 0)^T$,
2. (x^2, y^2), $x^2 = y^2 = (0, 0, 1, 0)^T$,
3. (x^3, y^3), $x^3 = y^3 = (\frac{1}{5}, \frac{2}{5}, 0, \frac{2}{5})^T$,

among others.

Support set invariancy – an example

Example (cont.)

1. Put $S_1 := \{1, 2, 3\}$ and $S_2 := \{1, 2, 3\}$.
   - (SSI1) doesn't hold;
   - (SSI2) one of sufficient conditions holds;
   - (SSI3) necessary condition doesn't hold.
2. Put $S_1 := \{3\}$ and $S_2 := \{3\}$.
   - (SSI1) holds;
   - (SSI2) sufficient conditions hold;
   - (SSI3) necessary condition holds.
3. Put $S_1 := \{1, 2, 4\}$ and $S_2 := \{1, 2, 4\}$.
   - (SSI1) doesn't hold;
   - (SSI2) no sufficient condition holds;
   - (SSI3) necessary condition doesn't hold.