Generalized linear fractional programming under interval uncertainty

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Introduction

Generalized linear fractional programming problem (GLFP)

- Let \( A, B \in \mathbb{R}^{m \times n}, \ C \in \mathbb{R}^{l \times n} \) and \( c \in \mathbb{R}^l \). Then (GLFP) is

  \[
  f(A, B, C, c) := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \ Cx \leq c, \ x \geq 0,
  \]
  
or,

  \[
  f(A, B, C, c) := \inf \left( \max_{i=1,\ldots,m} \frac{A_ix}{B_ix} \right) \text{ subject to } Cx \leq c, \ x \geq 0.
  \]

- Assume that \( Bx \geq 0 \) holds for all \( x \) satisfying \( Cx \leq c, \ x \geq 0 \).
- Solvable in polynomial time using an interior point method.

Interval problem

- An interval matrix \( A := [A, \bar{A}] = \{ A \in \mathbb{R}^{n \times n} \mid A \leq A \leq \bar{A} \} \).
- Analogously \( B, C \) and \( c \).
- Interval problem: (GLFP) with \( A \in A, B \in B, C \in C \) and \( c \in c \).
Assumption

\( (A1) \) For every \( B \in B \), \( C \in C \) and \( c \in c \) any solution to \( Cx \geq c \), \( x \geq 0 \) solves also \( Bx \geq 0 \).

Theorem

\( (A1) \) is true iff \( Bx \geq 0 \) holds for all \( x \) satisfying \( Cx \leq \bar{c} \), \( x \geq 0 \).

Definition (Bounds on optimal value)

Lower and upper bound on the optimal value is respectively defined as

\[
\underline{f} := \inf f(A, B, C, c) \text{ subject to } A \in A, \ B \in B, \ C \in C, \ c \in c,
\]

\[
\bar{f} := \sup f(A, B, C, c) \text{ subject to } A \in A, \ B \in B, \ C \in C, \ c \in c.
\]
Theorem

1. \textit{(Lower bound)} Let

\[ f_1 := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \; \lambda \leq 0, \; Cx \leq c, \; x \geq 0. \]

If \( f_1 < 0 \) then \( f = f_1 \), otherwise \( f = f_2 \) with

\[ f_2 := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \; \lambda \geq 0, \; Cx \leq c, \; x \geq 0. \]

2. \textit{(Upper bound)} Let

\[ f_3 := \inf \lambda \text{ subject to } \bar{A}x \leq \lambda Bx, \; \lambda \geq 0, \; \bar{C}x \leq c, \; x \geq 0. \]

If \( f_3 > 0 \) then \( \bar{f} = f_3 \), otherwise \( \bar{f} = f_4 \) with

\[ f_4 := \inf \lambda \text{ subject to } \bar{A}x \leq \lambda Bx, \; \lambda \leq 0, \; \bar{C}x \leq c, \; x \geq 0. \]
Tolerances of variations

Aim

- Given (GLFP)

\[ f(A, B, C, c) := \inf \lambda \text{ subject to } Ax \leq \lambda Bx, \ Cx \leq c, \ x \geq 0, \]

with \( A := A^0, \ B := B^0, \ C := C^0, \ c := c^0. \)

- Given bounds of the optimal value function \( f \) and \( \bar{f}, \)

\[ f \leq f(A^0, B^0, C^0, c^0) \leq \bar{f}. \]

- Compute maximal tolerances on inputs such that optimal values range in \([f, \bar{f}].\)

Tolerance rates

Given non-negative \( A^\Delta, B^\Delta \in \mathbb{R}^{m \times n}, \ C^\Delta \in \mathbb{R}^{l \times n} \) and \( c^\Delta \in \mathbb{R}^l. \) Denote

\[
A_\delta := [A^0 - \delta A^\Delta, A^0 + \delta A^\Delta], \quad B_\delta := [B^0 - \delta B^\Delta, B^0 + \delta B^\Delta], \\
C_\delta := [C^0 - \delta C^\Delta, C^0 + \delta C^\Delta], \quad c_\delta := [c^0 - \delta c^\Delta, c^0 + \delta c^\Delta].
\]
Tolerances of variations

Setting of tolerance rates

Put

- \( \Delta_{ij} := 0 \) if tolerance for \( a_{ij}^0 \) is not in demand;
- \( \Delta_{ij} := 1 \) for the absolute tolerance;
- \( \Delta_{ij} := |a_{ij}^0| \) for the relative (percentage) tolerance.

Tolerances

- A lower tolerance is \( \delta_1 > 0 \) such that for all \( A \in A_{\delta_1}, B \in B_{\delta_1}, C \in C_{\delta_1} \) and \( c \in c_{\delta_1} \):
  \[
  f(A, B, C, c) \geq f.
  \]
- An upper tolerance is \( \delta_1 > 0 \) such that for all \( A \in A_{\delta_2}, B \in B_{\delta_2}, C \in C_{\delta_2} \) and \( c \in c_{\delta_2} \):
  \[
  f(A, B, C, c) \leq \bar{f}.
  \]
- A (overall) tolerance \( \delta = \min(\delta_1, \delta_2) \).
Tolerances of variations

Lemma

Let

\[ \delta_1^* := \sup \delta \]

subject to \( f \leq f(A, B, C, c) \) \( \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta, \)

\[ \delta_2^* := \sup \delta \]

subject to \( \bar{f} \geq f(A, B, C, c) \) \( \forall A \in A_\delta, B \in B_\delta, C \in C_\delta, c \in c_\delta, \)

and denote \( \delta^* = \min(\delta_1^*, \delta_2^*) \). Assume that

(A2) \( (B^0 - \delta^* B^\Delta)x > 0 \) for all solutions of \( (C^0 - \delta^* C^\Delta)x \leq c + \delta^* c^\Delta, \)
Kid x \geq 0.

(A3) \( (C^0 + \delta^* C^\Delta)x \leq c - \delta^* c^\Delta, \) \( x \geq 0 \) is solvable.

Then \( \delta^* \) is the maximal admissible tolerance.
Tolerances of variations

Example ((A2) is necessary)
Consider the problem

\[ \inf \lambda \ \text{subject to} \ [1 - \delta, 1 + \delta]x \leq [1 - \delta, 1 + \delta]\lambda x, \ x \geq 1, \]

- For \( \delta \in (0, 1) \) the optimal value ranges in \( \left[ \frac{1-\delta}{1+\delta}, \frac{1+\delta}{1-\delta} \right] \),
- For \( \delta = 1 \) the optimal value can achieve \(-\infty\).

Example ((A3) is necessary)
Consider the problem

\[ \inf \lambda \ \text{s.t.} \ x_1 + x_2 \leq \lambda(x_1 + x_2), \ x_2 = 1, \ [1 - \delta, 1 + \delta]x_1 + x_2 \geq 2, \]

- For \( \delta \in (0, 1) \) the optimal value is constantly one,
- For \( \delta = 1 \) the optimal value is either one or \( \infty \).
Tolerances of variations

**Theorem (Tolerances of variations)**

Under assumption (A2) and (A3):

1. **(Lower tolerance)** If \( f \geq 0 \) then

\[
\delta_1 := \inf \delta \quad \text{subject to} \quad (A^0 - fB^0)x \leq \delta(A^\Delta + fB^\Delta)x, \\
C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), \ x \geq 0,
\]

otherwise

\[
\delta_1 := \inf \delta \quad \text{subject to} \quad (A^0 - fB^0)x \leq \delta(A^\Delta - fB^\Delta)x, \\
C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), \ x \geq 0,
\]

2. **(Upper tolerance)** If \( \overline{f} \geq 0 \) then

\[
\delta_2 := \sup \delta \quad \text{subject to} \quad (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta + \overline{f}B^\Delta)x, \\
-C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0,
\]

otherwise

\[
\delta_2 := \sup \delta \quad \text{subject to} \quad (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta - \overline{f}B^\Delta)x, \\
-C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0.
\]
Tolerances of variations

Remark

- The resulting tolerances $\delta_1$ and $\delta_2$ are maximal in the most of cases.
- Compute $\delta_1$ and $\delta_2$ and then check validity of assumptions (A2) and (A3) with $\delta^* = \min(\delta_1, \delta_2)$.

Example (von Neumann economic growth model)

Consider

$$\max \lambda \quad \text{subject to} \quad \lambda Ax \leq Bx, \ x \geq 1,$$

where

- variables $x_i$, $i = 1, \ldots, n$ denote activity of sector $i$;
- matrix $A \in \mathbb{R}^{m \times n}$ consists of input coefficients;
- matrix $B \in \mathbb{R}^{m \times n}$ consists of output coefficients.
Consider data set by Li (2008):

\[
A = \begin{pmatrix}
0.28 & 0.50 & 0.53 & 0 & 0 & 0 \\
0.84 & 0 & 0 & 0 & 0 & 0.77 \\
0 & 0.49 & 0.45 & 0.50 & 0.48 & 0 \\
0 & 0 & 0 & 0.51 & 0.57 & 0.29
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0.25 & 1 & 1 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The optimal value is \( \lambda^* = 1.049 \).

1. Put \([\underline{f}, \bar{f}] := [1, 1.2] \) and \( A^\Delta := |A| \) and \( B^\Delta := |B| \).
Compute lower tolerance \( \delta^1 = 0.024 \) and upper tolerance \( \delta^2 = 0.067 \).
Entries of \( A \) and \( B \) may vary within 2.4% tolerance.

2. Put \([\underline{f}, \bar{f}] := [1, 1.2] \) and \( A^\Delta := 0 \) and \( B^\Delta := |B| \).
Compute \( \delta^1 = 0.046 \) and \( \delta^2 = 0.143 \).
The resulting percentage tolerance is 4.6%.
The End.