# Generalized linear fractional programming under interval uncertainty

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# Introduction

## Generalized linear fractional programming problem (GLFP)

- Let $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{l\times n}$ and $c \in \mathbb{R}^{l}$. Then (GLFP) is

$$f(A,B,C,c) := \inf\ \lambda \ \text{ subject to } \ Ax \le \lambda Bx,\ \ Cx \le c,\ \ x \ge 0,$$

  or,

$$f(A,B,C,c) := \inf\ \left(\max_{i=1,\dots,m} \frac{A_i x}{B_i x}\right) \ \text{ subject to } \ Cx \le c,\ \ x \ge 0.$$

- Assume that $Bx \ge 0$ holds for all $x$ satisfying $Cx \le c$, $x \ge 0$.
- Solvable in polynomial time using an interior point method.

## Interval problem

- An interval matrix $\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n} \mid \underline{A} \le A \le \overline{A}\}$. Analogously $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{c}$.
- Interval problem: (GLFP) with $A \in \mathbf{A}$, $B \in \mathbf{B}$, $C \in \mathbf{C}$ and $c \in \mathbf{c}$.

# Introduction

## Assumption

(A1) For every $B \in \mathbf{B}$, $C \in \mathbf{C}$ and $c \in \mathbf{c}$ any solution to $Cx \geq c$, $x \geq 0$ solves also $Bx \geq 0$.

## Theorem

(A1) *is true iff* $\underline{B}x \geq 0$ *holds for all* $x$ *satisfying* $\underline{C}x \leq \overline{c}$, $x \geq 0$.

## Definition (Bounds on optimal value)

Lower and upper bound on the optimal value is respectively defined as

$$\underline{f} := \inf \; f(A, B, C, c) \;\; \text{subject to} \;\; A \in \mathbf{A},\; B \in \mathbf{B},\; C \in \mathbf{C},\; c \in \mathbf{c},$$
$$\overline{f} := \sup \; f(A, B, C, c) \;\; \text{subject to} \;\; A \in \mathbf{A},\; B \in \mathbf{B},\; C \in \mathbf{C},\; c \in \mathbf{c}.$$

# Range of optimal values

## Theorem

1. *(Lower bound) Let*

$$f_1 := \inf\ \lambda \ \text{ subject to } \ \underline{A}x \le \lambda \underline{B}x,\ \lambda \le 0,\ \underline{C}x \le \overline{c},\ x \ge 0.$$

*If* $f_1 < 0$ *then* $\underline{f} = f_1$, *otherwise* $\underline{f} = f_2$ *with*

$$f_2 := \inf\ \lambda \ \text{ subject to } \ \underline{A}x \le \lambda \overline{B}x,\ \lambda \ge 0,\ \underline{C}x \le \overline{c},\ x \ge 0.$$

2. *(Upper bound) Let*

$$f_3 := \inf\ \lambda \ \text{ subject to } \ \overline{A}x \le \lambda \underline{B}x,\ \lambda \ge 0,\ \overline{C}x \le \underline{c},\ x \ge 0.$$

*If* $f_3 > 0$ *then* $\overline{f} = f_3$, *otherwise* $\overline{f} = f_4$ *with*

$$f_4 := \inf\ \lambda \ \text{ subject to } \ \overline{A}x \le \lambda \overline{B}x,\ \lambda \le 0,\ \overline{C}x \le \underline{c},\ x \ge 0.$$

# Tolerances of variations

## Aim

- Given (GLFP)

$$f(A,B,C,c) := \inf\ \lambda \ \text{ subject to }\ Ax \le \lambda Bx,\ Cx \le c,\ x \ge 0,$$

  with $A := A^0,\ B := B^0,\ C := C^0,\ c := c^0$.
- Given bounds of the optimal value function $\underline{f}$ and $\overline{f}$,
  $\underline{f} \le f(A^0, B^0, C^0, c^0) \le \overline{f}$.
- Compute maximal tolerances on inputs such that optimal values range in $[\underline{f}, \overline{f}]$.

## Tolerance rates

Given non-negative $A^\Delta, B^\Delta \in \mathbb{R}^{m\times n}$, $C^\Delta \in \mathbb{R}^{l\times n}$ and $c^\Delta \in \mathbb{R}^l$. Denote

$$\mathbf{A}_\delta := [A^0 - \delta A^\Delta, A^0 + \delta A^\Delta],\quad \mathbf{B}_\delta := [B^0 - \delta B^\Delta, B^0 + \delta B^\Delta],$$
$$\mathbf{C}_\delta := [C^0 - \delta C^\Delta, C^0 + \delta C^\Delta],\quad \mathbf{c}_\delta := [c^0 - \delta c^\Delta, c^0 + \delta c^\Delta].$$

# Tolerances of variations

## Setting of tolerance rates

Put

- $a_{ij}^{\Delta} := 0$ if tolerance for $a_{ij}^{0}$ is not in demand;
- $a_{ij}^{\Delta} := 1$ for the absolute tolerance;
- $a_{ij}^{\Delta} := |a_{ij}^{0}|$ for the relative (percentage) tolerance.

## Tolerances

- A lower tolerance is $\delta_1 > 0$ such that for all $A \in \mathbf{A}_{\delta_1}$, $B \in \mathbf{B}_{\delta_1}$, $C \in \mathbf{C}_{\delta_1}$ and $c \in \mathbf{c}_{\delta_1}$:
$$f(A, B, C, c) \geq \underline{f}.$$
- An upper tolerance is $\delta_1 > 0$ such that for all $A \in \mathbf{A}_{\delta_2}$, $B \in \mathbf{B}_{\delta_2}$, $C \in \mathbf{C}_{\delta_2}$ and $c \in \mathbf{c}_{\delta_2}$:
$$f(A, B, C, c) \leq \overline{f}.$$
- A (overall) tolerance $\delta = \min(\delta_1, \delta_2)$.

# Tolerances of variations

## Lemma

*Let*

$$\delta_1^* := \sup \ \delta$$
$$\textit{subject to} \ \ \underline{f} \le f(A,B,C,c) \ \forall A \in \mathbf{A}_\delta,\ B \in \mathbf{B}_\delta,\ C \in \mathbf{C}_\delta,\ c \in \mathbf{c}_\delta,$$
$$\delta_2^* := \sup \ \delta$$
$$\textit{subject to} \ \ \overline{f} \ge f(A,B,C,c) \ \forall A \in \mathbf{A}_\delta,\ B \in \mathbf{B}_\delta,\ C \in \mathbf{C}_\delta,\ c \in \mathbf{c}_\delta,$$

*and denote* $\delta^* = \min(\delta_1^*, \delta_2^*)$. *Assume that*

(A2) $(B^0 - \delta^* B^\Delta)x > 0$ *for all solutions of* $(C^0 - \delta^* C^\Delta)x \le c + \delta^* c^\Delta$, $x \ge 0$.

(A3) $(C^0 + \delta^* C^\Delta)x \le c - \delta^* c^\Delta$, $x \ge 0$ *is solvable.*

*Then* $\delta^*$ *is the maximal admissible tolerance.*

# Tolerances of variations

## Example ((A2) is necessary)

Consider the problem

$$\inf\ \lambda \ \text{ subject to }\ [1-\delta, 1+\delta]x \le [1-\delta, 1+\delta]\lambda x,\ x \ge 1,$$

- For $\delta \in (0,1)$ the optimal value ranges in $[\frac{1-\delta}{1+\delta}, \frac{1+\delta}{1-\delta}]$,
- For $\delta = 1$ the optimal value can achieve $-\infty$.

## Example ((A3) is necessary)

Consider the problem

$$\inf\ \lambda \ \text{ s.t. }\ x_1 + x_2 \le \lambda(x_1 + x_2),\ x_2 = 1,\ [1-\delta, 1+\delta]x_1 + x_2 \ge 2,$$

- For $\delta \in (0,1)$ the optimal value is constantly one,
- For $\delta = 1$ the optimal value is either one or $\infty$.

## Theorem (Tolerances of variations)

*Under assumption (A2) and (A3):*

1. *(Lower tolerance) If* $\underline{f} \geq 0$ *then*

$$\delta_1 := \inf \ \delta \ \ \textit{subject to} \ \ (A^0 - \underline{f}B^0)x \leq \delta(A^\Delta + \underline{f}B^\Delta)x, \\ C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), \ x \geq 0,$$

   *otherwise*

$$\delta_1 := \inf \ \delta \ \ \textit{subject to} \ \ (A^0 - \underline{f}B^0)x \leq \delta(A^\Delta - \underline{f}B^\Delta)x, \\ C^0x - c^0 \leq \delta(C^\Delta x + c^\Delta), \ x \geq 0.$$

2. *(Upper tolerance) If* $\overline{f} \geq 0$ *then*

$$\delta_2 := \sup \ \delta \ \ \textit{subject to} \ \ (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta + \overline{f}B^\Delta)x, \\ -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0,$$

   *otherwise*

$$\delta_2 := \sup \ \delta \ \ \textit{subject to} \ \ (-A^0 + \overline{f}B^0)x \geq \delta(A^\Delta - \overline{f}B^\Delta)x, \\ -C^0x + c^0 \geq \delta(C^\Delta x + c^\Delta), \ x \geq 0.$$

# Tolerances of variations

## Remark

- The resulting tolerances $\delta_1$ and $\delta_2$ are maximal in the most of cases.
- Compute $\delta_1$ and $\delta_2$ and then check validity of assumptions (A2) and (A3) with $\delta^* = \min(\delta_1, \delta_2)$.

## Example (von Neumann economic growth model)

Consider

$$\max\ \lambda \ \text{ subject to } \ \lambda A x \leq B x,\ x \geq 1,$$

where

- variables $x_i$, $i = 1, \ldots, n$ denote activity of sector $i$;
- matrix $A \in \mathbb{R}^{m \times n}$ consists of input coefficients;
- matrix $B \in \mathbb{R}^{m \times n}$ consists of output coefficients.

# Tolerances of variations

## Example (cont.)

Consider data set by Li (2008):

$$A = \begin{pmatrix} 0.28 & 0.50 & 0.53 & 0 & 0 & 0 \\ 0.84 & 0 & 0 & 0 & 0 & 0.77 \\ 0 & 0.49 & 0.45 & 0.50 & 0.48 & 0 \\ 0 & 0 & 0 & 0.51 & 0.57 & 0.29 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0.25 & 1 & 1 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The optimal value is $\lambda^* = 1.049$.

1. Put $[\underline{f}, \overline{f}] := [1, 1.2]$ and $A^\Delta := |A|$ and $B^\Delta := |B|$.
   Compute lower tolerance $\delta^1 = 0.024$ and upper tolerance $\delta^2 = 0.067$.
   Entries of $A$ and $B$ may vary within 2.4% tolerance.
2. Put $[\underline{f}, \overline{f}] := [1, 1.2]$ and $A^\Delta := 0$ and $B^\Delta := |B|$.
   Compute $\delta^1 = 0.046$ and $\delta^2 = 0.143$.
   The resulting percentage tolerance is 4.6%.

The End.