Tolerance analysis in linear programming

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Introduction

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- An inequality system $Dx \leq d$.
- An interior point $x^* \in \mathbb{R}^n$.

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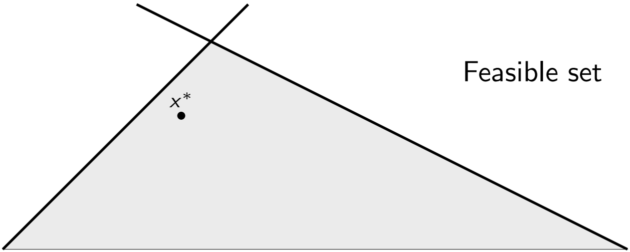

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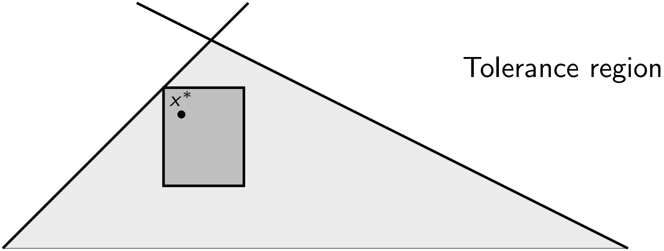

Wendell's symmetric tolerances

Definition

Let $x^{\Delta} \in \mathbb{R}^n_+$. Then δ^* is *symmetric tolerance* if $Dx \leq d$ holds for every x such that $|x_j - x_j^*| \leq \delta^* x_j^{\Delta}$, $j = 1, \ldots, n$.

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Theorem (Wendell, 1984)

We have

$$\delta^* = \inf_{i=1,\ldots,m;\ |D_{i\cdot}|x^{\Delta}>0} \frac{d_i - D_{i\cdot}x^*}{|D_{i\cdot}|x^{\Delta}}.$$

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Properties

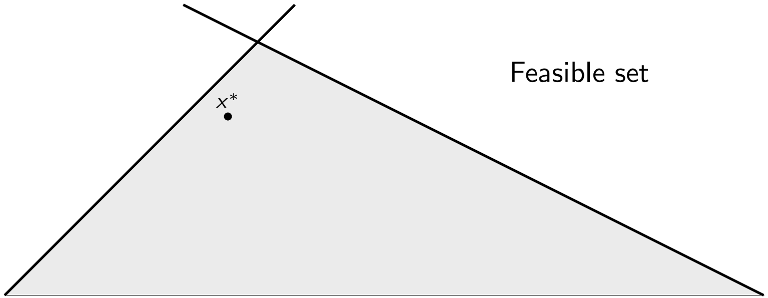
- Easy to compute and interpret
- Maximal symmetric tolerance
- But: small and loss of information

Example

Consider the inequality system

$$-x_1 + x_2 \leq 2,$$
$$x_1 + 2x_2 \leq 12,$$

and an initial feasible point $x^* = (2,3)^T$. Let $x^\Delta = |x^*|$.



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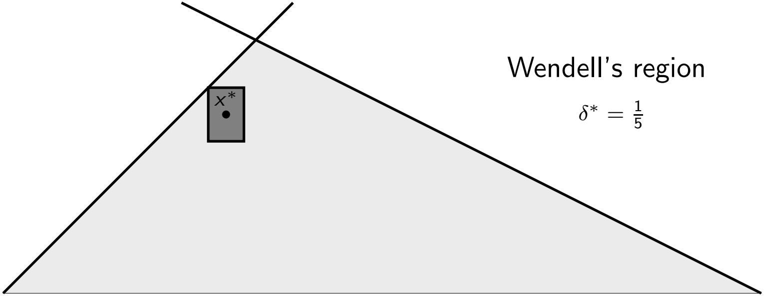

Wondolowski's individual tolerances

Definition

Let $x^{\Delta} \in \mathbb{R}^n_+$. Then $\delta^-, \delta^+ \in \mathbb{R}^n_+$ are *tolerances* if $Dx \leq d$ holds for every for very x such that $x^*_j - \delta^-_j x^{\Delta}_j \leq x_j \leq x^*_j + \delta^+_j x^{\Delta}_j$, $j = 1, \ldots, n$.

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Properties

- Easy to compute and interpret
- Not maximal

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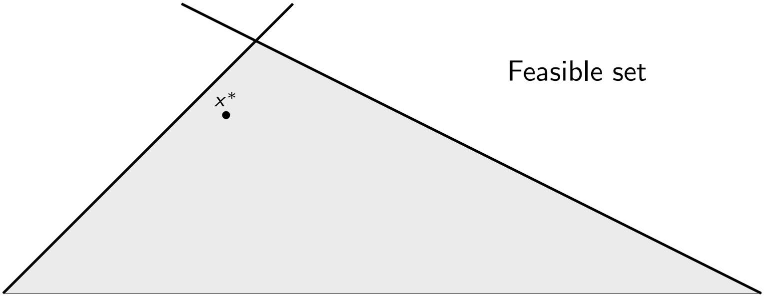

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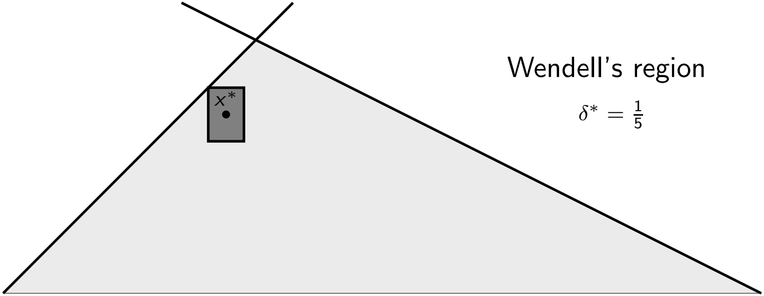

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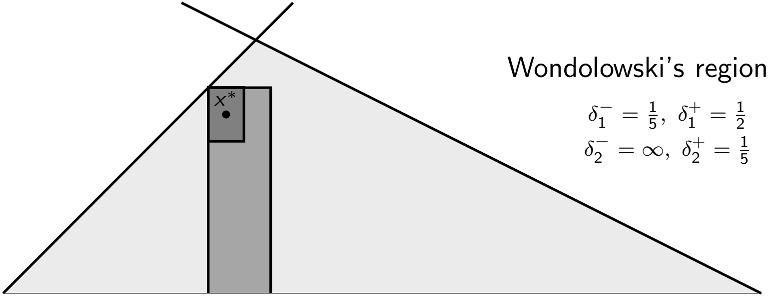

Wondolowski's region

$\delta_1^- = \frac{1}{5}, \ \delta_1^+ = \frac{1}{2}$
$\delta_2^- = \infty, \ \delta_2^+ = \frac{1}{5}$

Optimal individual tolerances

Basic idea

- Repeat Wondolowski's method until all inequalities are fill up

Properties

- Number of iterations at most $\min(m, 2n)$
- Tolerances are maximal and proportionally determined

Optimal individual tolerances – an algorithm

1. $I := \{1, \dots, m\}$, $\alpha_j^+ := 1$, $\alpha_j^- := 1$ $\forall j = 1, \dots, n$;
2. **while** $I \neq \emptyset$ and $\exists j : (\alpha_j^+ = 1$ or $\alpha_j^- = 1)$ **do**
3. $\quad R_i := d_i - D_{i\cdot} x^* - \sum_{j;\ d_{ij}>0,\ \alpha_j^+=0} d_{ik} x_j^\Delta \delta_j^+ + \sum_{j;\ d_{ij}<0,\ \alpha_j^+=0} d_{ik} x_j^\Delta \delta_j^-, \ \forall i \in I;$
4. $\quad S_i := \sum_{j;\ d_{ij}>0,\ \alpha_j^+=1} d_{ij} x_j^\Delta - \sum_{j;\ d_{ij}<0,\ \alpha_j^-=1} d_{ij} x_j^\Delta, \ \forall i \in I;$
5. $\quad$ **for all** $j \in \{1, \dots, n\}$ **do**
6. $\qquad$ **if** $\alpha_j^+ = 1$ **then** $\delta_j^+ := \inf_{i \in I;\ S_i > 0,\ d_{ij}>0} \dfrac{R_i}{S_i}$;
7. $\qquad$ **if** $\alpha_j^- = 1$ **then** $\delta_j^- := \inf_{i \in I;\ S_i > 0,\ d_{ij}<0} \dfrac{R_i}{S_i}$;
8. $\quad$ **for all** $i \in I$ **do**
9. $\qquad$ **if** $D_{i\cdot} x^* + \sum_{k;\ d_{ik}>0} d_{ik} x_k^\Delta \delta_k^+ + \sum_{k;\ d_{ik}<0} d_{ik} x_k^\Delta \delta_k^- = d_i$ **then**
10. $\qquad\quad I := I \setminus \{i\}$;
11. $\qquad\quad$ **for all** $j \in \{1, \dots, n\}$ **do**
12. $\qquad\qquad$ **if** $d_{ij} > 0$ **then** $\alpha_j^+ := 0$;
13. $\qquad\qquad$ **if** $d_{ij} < 0$ **then** $\alpha_j^- := 0$;

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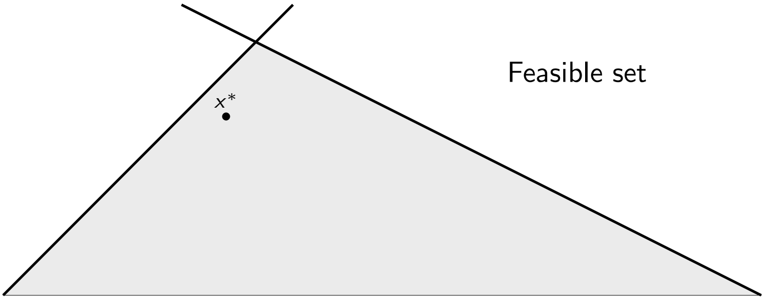

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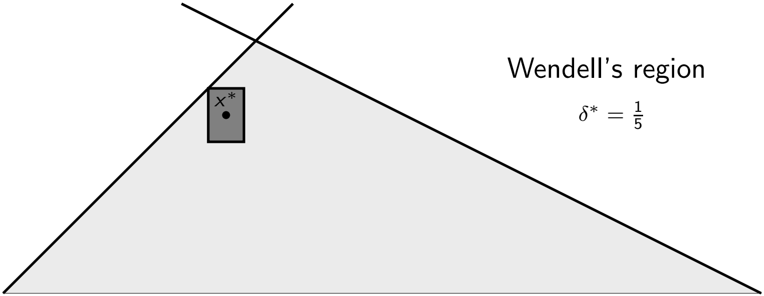

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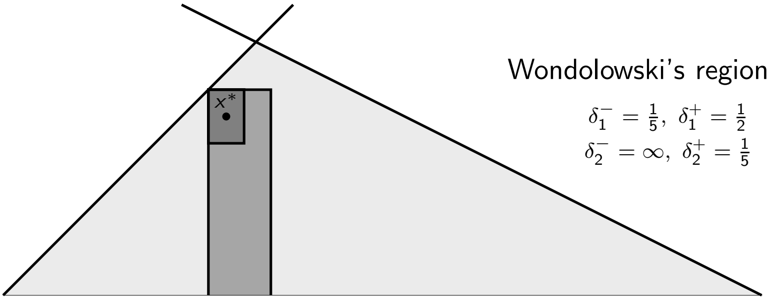

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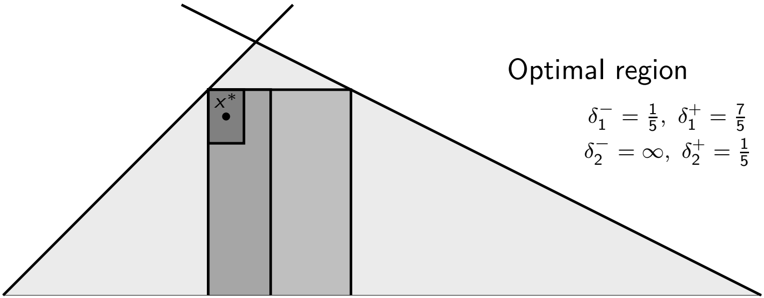

Linear programming issues

Consider a linear programming problem

$$\min \; c^T x \;\text{ subject to }\; Ax = b, \; x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Let x^* be an optimal solution.

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Invariacies

- optimal basis invariancy
- support set invariancy
- optimal partition invariancy

Objective function coefficients

Optimal basis invariancy

Let $B \subseteq \{1, \dots, n\}$ be an optimal basis and $N := \{1, \dots, n\} \setminus B$.

Invariancy: How much can c perturb such that B remains optimal basis.

The invariancy region is described by

$$c_N - (A_B^{-1} A_N)^T c_B \geq 0.$$

- It is linear inequality system w.r.t. variables c
- Tolerance approach can be directly applied
- Related to simplex method

Objective function coefficients

Support set invariancy

Support for x^* is defined as $\sigma(x^*) := \{i \mid x_i^* > 0\}$.

Invariancy: How much can c perturb such that there is optimal solution x^0 such that $\sigma(x^0) = \sigma(x^*)$.

Objective function coefficients

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Optimal partition invariancy

Optimal partition of $\{1, \ldots, n\}$ into disjoint sets

$$\mathcal{B} := \{i \mid x_i > 0 \text{ for some optimal } x\},$$
$$\mathcal{N} := \{i \mid c_i - A_{\cdot i}^T y > 0 \text{ for some dual optimal } y\}.$$

Invariancy: How much can c perturb such that the optimal partition remains the same.

Objective function coefficients

Support set and optimal partition invariancy

- Denote $Z := \{1, \ldots, n\} \setminus \sigma(x^*)$
- Suppose that $\{x \mid Ax = 0, x_Z = 0\} = \{0\}$
- Let g_k, $k \in K$, be all extremal directions of

$$\{x \mid Ax = 0, x_Z \geq 0\}.$$

Objective function coefficients

Support set and optimal partition invariancy

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The support set invariancy region is described by

$$g_k^T c \geq 0, \quad k \in K,$$

Objective function coefficients

Support set and optimal partition invariancy

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- Let g_k, $k \in K$, be all extremal directions of

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The support set invariancy region is described by

$$g_k^T c \geq 0, \quad k \in K,$$

The optimal partition invariancy region is described by

$$g_k^T c > 0, \quad k \in K,$$

as long as x^* is a strictly complementary solution (i.e., $\sigma(x^* = \mathcal{B})$).

Objective function coefficients

Properties

- It is linear inequality system w.r.t. variables c
- Tolerance approach can be directly applied
- Support set invariancy region is maximal region where x^* remains optimal

Theorem (Hladík, 2008)

Let $x^{\Delta} := e$ *and* δ^* *be the maximal (Wendell's) symmetric tolerance for the support set invariancy. Then checking whether* $\delta^* \leq 1$ *is NP-hard.*

Objective function coefficients

Sketch of Proof.

Maximal tolerance δ^* is computed by

$$\max\ \delta \ \text{ subject to } \ \gamma^T x^* \le \gamma^T x \ \ \forall x \in \mathcal{X}\ \forall \gamma : |\gamma - c| \le \delta e,$$

or, equivalently

$$\inf\ \delta \ \text{ subject to } \ \gamma^T x^* > \gamma^T x,\ x \in \mathcal{X},\ |\gamma - c| \le \delta e.$$

Substitute $z := x - x^*$ and simplify to

$$\inf\ \delta \ \text{ subject to } \ c^T z - \delta e^T |z| < 0,\ Az = b - Ax^*,\ z \ge -x^*.$$

Construct a polynomial reduction from the NP-hard problem of testing solvability of

$$|Mx| \le e,\ e^T |x| > 1.$$

Right-hand side coefficients

Optimal basis invariancy

Let $B \subseteq \{1, \dots, n\}$ be optimal basis. Then the invariancy region is described by

$$A_B^{-1} b \geq 0.$$

Support set and optimal partition invariancy

Let $P := \sigma(x^*)$ and suppose that $\{y \mid A_P^T y = 0\} = \{0\}$.
Let h_k, $k \in K'$, are all extremal directions of $\{y \mid A_P^T y \leq 0\}$.
Then the support set invariancy region is described by

$$h_k^T b < 0, \quad k \in K'.$$

The optimal partition invariancy region is the same as long as x^* is strictly complementary optimal solution.

Linear programming issues

Conclusion

Optimal basis vs. support set and optimal partition invariancy:

- Objective function coefficients:
  - Optimal basis invariancy region is smaller.
    (particularly for degenerate optimal solution)
- Right-hand side coefficients:
  - Optimal basis invariancy region is larger.
    (particularly for degenerate optimal solution)
  - But: Optimal basis invariancy applicable only for basic solutions.

Last slide

The End.