Computing the range of real eigenvalues of an interval matrix

Milan Hladík & David Daney

INRIA, project Coprin
Sophia Antipolis, France

SWIM 08, Montpellier, France
June 19–20, 2008

Introduction

Notation

An interval matrix

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n \times n} \mid \underline{A} \leq A \leq \overline{A}\},$$

Midpoint and radius of $\mathbf{A}$

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Aim

Determine eigenvalue set

$$\Lambda := \{\lambda \in \mathbb{R};\ Ax = \lambda x,\ x \neq 0,\ A \in \mathbf{A}\}$$

Complexity

Checking $0 \in \Lambda$ is NP-hard.

Algorithm

Algorithm (Approximation of Λ)

1. Compute initial bounds $\underline{\lambda}^0$, $\overline{\lambda}^0$ such that $\Lambda \subseteq \boldsymbol{\lambda}^\mathbf{0} := [\underline{\lambda}^0, \overline{\lambda}^0]$;
2. $L := \{\boldsymbol{\lambda}^\mathbf{0}\}$, $L_{\text{inn}} := \emptyset$, $L_{\text{unc}} := \emptyset$;
3. while $L \neq \emptyset$ do
   1. choose and remove some $\boldsymbol{\lambda}$ from L;
   2. (Outer test)
      if $\boldsymbol{\lambda} \cap \Lambda = \emptyset$ then loop;
   3. (Inner test)
      if $\boldsymbol{\lambda} \subseteq \Lambda$ then $L_{\text{inn}} := L_{\text{inn}} \cup \{\lambda\}$ and loop;
   4. (Precision test)
      if $\lambda_\Delta < \varepsilon$ then $L_{\text{unc}} := L_{\text{unc}} \cup \{\lambda\}$ and loop;
   5. (Division)
      $\boldsymbol{\lambda}^1 := [\underline{\lambda}, \lambda_c]$, $\boldsymbol{\lambda}^2 := [\lambda_c, \overline{\lambda}]$, $L := L \cup \{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2\}$ and loop;
4. return L_{inn} as inner and L_{unc} as unrecognized intervals;

Algorithm

Implementation of the list L

Implemented as a stack (LIFO), useful properties:

- neighbor of an inner box cannot be outer
- neighbor of an outer box cannot be inner

Initial bounds (Rohn, 1998)

Let

$$S_c := \frac{1}{2}\left(A_c + A_c^T\right), \quad S_\Delta := \frac{1}{2}\left(A_\Delta + A_\Delta^T\right).$$

Then $\Lambda \subseteq \boldsymbol{\lambda}^0 := [\underline{\lambda}^0, \overline{\lambda}^0]$, where

$$\begin{aligned} \underline{\lambda}^0 &= \lambda_{\min}(S_c) - \rho(S_\Delta), \\ \overline{\lambda}^0 &= \lambda_{\max}(S_c) + \rho(S_\Delta). \end{aligned}$$

Outer test

Remark

λ is eigenvalue of A iff $A - \lambda I$ is singular.

Proposition

If the interval matrix $\mathbf{A} - \boldsymbol{\lambda} I$ is regular, then $\boldsymbol{\lambda}$ is an outer interval.

Idea

Put $\mathbf{M} := \mathbf{A} - \boldsymbol{\lambda} I$ and check regularity of $\mathbf{M}$.

Theorem (Sufficient regularity condition)

An interval matrix $\mathbf{M}$ is regular if M_c is nonsingular and $\rho(|M_c^{-1}|M_\Delta) < 1$.

Outer test: Jansson & Rohn method

Theorem (Jansson & Rohn, 1999)

Let $b \in \mathbb{R}^n$ and consider $\mathbf{M}x = b$. The solution set is described by

$$|M_c x - b| \leq M_\Delta |x|.$$

Then $\mathbf{M}$ is regular if and only if its any component is bounded.

Algorithm (Basic scheme)

1. Select $b \in \mathbb{R}^n$ and the initial orthant;
2. Check for unboundedness. If so, then $\mathbf{M}$ is not regular;
3. Repeat for the neighboring orthants, and so on;
4. If inspected the whole component, then $\mathbf{M}$ is regular.

Properties

- Not necessarily exponential complexity.

Outer test: ILS method

Proposition

M *is regular iff there is no solution of the interval system*

$$\mathbf{M}x = 0, \ \|x\|_\infty = 1.$$

Algorithm (ILS method)

1. For $i = 1, \ldots, n$ do
    1. $\mathbf{b} :=$ (the i-th column of $\mathbf{M}$), $\mathbf{M}' :=$ ($\mathbf{M}$ without the i-th column);
    2. Solve (approximately) the interval linear system
    $$\mathbf{M}'x' = -\mathbf{b}, \ -e \le x' \le e;$$
    3. If it has possibly a solution, then stop, **M** needn't be regular;
2. Conclude that **M** is regular.

Properties

- We obtain also estimation of eigenvectors.

Inner test

Algorithm (Inner test for $\boldsymbol{\lambda}$)

1. Call Jansson & Rohn algorithm with $\mathbf{M} := \mathbf{A} - \lambda_c I$;
2. If $\mathbf{M}$ is regular then stop, $\boldsymbol{\lambda}$ is not inner interval;
3. else, let $z \in \{\pm 1\}^n$ be a sign vector for the "unbounded orthant";
4. Solve linear program

$$\max\left\{z^T x^1 - z^T x^2;\ (A_c - A_\Delta \operatorname{diag}(z))(x^1 - x^2) - \underline{\lambda} x^1 + \overline{\lambda} x^2 \leq b,\right.$$
$$(A_c + A_\Delta \operatorname{diag}(z))(x^1 - x^2) - \overline{\lambda} x^1 + \underline{\lambda} x^2 \geq b,$$
$$\left.\operatorname{diag}(z)(x^1 - x^2) \geq 0,\ x^1, x^2 \geq 0\right\}$$

5. If it is unbounded then $\boldsymbol{\lambda}$ is an inner interval.

Improvements

- Remember the sign vector z for the following test.

Exact bounds

Theorem (Rohn, 1993)

Let $\lambda \in \partial\Lambda$. Then there are $x, p \in \mathbb{R}^n \setminus \{0\}$ and $y, z \in \{\pm 1\}^n$ such that

$$\big(A_c - \operatorname{diag}(y)\, A_\Delta \operatorname{diag}(z)\,\big)x = \lambda x,$$
$$\big(A_c^T - \operatorname{diag}(z)\, A_\Delta^T \operatorname{diag}(y)\,\big)p = \lambda p,$$
$$\operatorname{diag}(z)\, x \geq 0,\ \operatorname{diag}(y)\, p \geq 0.$$

Algorithm (Exact bound candidates for $\boldsymbol{\lambda}$)

1. Call ILS method with the input matrix $\mathbf{A} - \boldsymbol{\lambda} I$ to obtain an outer approximation $\mathbf{x}$ of the right eigenvectors;
2. Call ILS method with the input matrix $(\mathbf{A} - \boldsymbol{\lambda} I)^T$ to obtain an outer approximation $\mathbf{p}$ of the left eigenvectors;
3. If the number of possible signs of $\mathbf{x}$ and $\mathbf{p}$ is small then enumerate all possibilities.

Numerical experiments

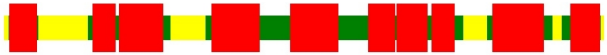
n	ε	R	time
5	0.1	1	2 sec
10	0.1	0.5	9 sec
10	0.1	5	1 min 12 sec
15	0.1	0.1	37 sec
15	0.1	0.5	10 min 29 sec
15	0.1	1	7 min 59 sec
20	0.1	0.1	2 min 16 sec
20	0.1	0.5	21 min 6 sec
25	0.1	0.01	5 min 46 sec
25	0.1	0.05	10 min 39 sec
30	0.01	0.01	14 min 37 sec
30	0.01	0.1	48 min 31 sec

- n=matrix dimension
- ε=accuracy
- random matrices **A**
- entries of A_c in uniform distribution $[-20, 20]$
- entries of A_Δ in uniform distribution $[0, R]$
- exact bounds found
- not rigorous:
  `GLPK` (linear programs)
  `CLAPACK` (linear algebra)

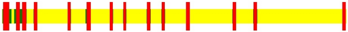
Numerical experiments

- red $=$ eigenvalue set Λ,
- yellow $=$ sufficient outer test condition,
- green $=$ Jansson & Rohn method

Random matrix,
$n = 30,\ R = 0.1$,
computing time 48 min 31 sec.



Random symmetric matrix,
$n = 15,\ R = 0.5$,
computing time 13 min 48 sec.



Random $\mathbf{A}^T\mathbf{A}$ matrix,
$n = 15,\ R = 0.02$,
computing time 1 min 58 sec.

Random nonnegative matrix,
$n = 15,\ R = 0.2$,
computing time 2 min 22 sec.

Last slide

Other approaches

- Interval analysis techniques (filtering, branch & prune, ...) for

$$\mathbf{A}x = \lambda x, \ \|x\|_\infty = 1, \ \lambda \in \boldsymbol{\lambda}^0.$$

  Efficient for $n < 5$.

- Gerschorin discs, Cassini ovals, ... no sharp results.

Present and future work

- Eigenvalues of symmetric interval matrix;
- Singular values of interval matrix;
- Complex eigenvalues of interval matrix.