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Tolerances in Portfolio Selection via Interval Linear Programming

Abstract

We consider a linear programming problem and develop an effective method for computing tolerances for input data. The tolerances are determined such that the input quantities can simultaneously and independently vary within these tolerances while the optimal value does not exceed given lower and upper bounds. In our approach we are able to take into account all the input quantities or some selected ones. The procedure runs in polynomial time. Although the tolerances are not the best possible (due to dependencies between quantities) in general, the results are satisfactory. We illustrate the procedure on a simple portfolio selection problem modelled as a linear program.

Keywords : *Portfolio selection, linear programming, generalized fractional programming, tolerance analysis, interval analysis.*

1 Introduction

The aim of this paper is to develop a tool for computing tolerances of possibly all input quantities of a linear programming problem. Our approach is based on interval analysis [1], so we first introduce some basic notations and theorems.

An interval matrix is defined as

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n};\ \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}$, $\overline{A} \in \mathbb{R}^{m \times n}$, $\underline{A} \leq \overline{A}$, are given matrices. By

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the midpoint and the radius of $\boldsymbol{A}$, respectively. Interval vectors are defined analogously.

Let an interval matrix $\boldsymbol{A} \subset \mathbb{R}^{m \times n}$ and interval vectors $\boldsymbol{b} \subset \mathbb{R}^m$, $\boldsymbol{c} \subset \mathbb{R}^n$ be given. By an interval linear programming problem we understand a family of problems

$$\min c^T x \ \text{ subject to } \ Ax \geq b, x \geq 0, \tag{1}$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$. Interval linear programs of this or similar type were studied e.g. in [2], [3], [6], [8], [11]. Denote by

$$f(A, b, c) := \inf c^T x \quad \text{subject to} \quad Ax \geq b, x \geq 0$$

the optimal value of the linear program, also infinite values are possible. The lower and upper bound of the optimal value is denoted, respectively, by

$$\begin{aligned} \underline{f} &:= \inf f(A, b, c) \quad \text{subject to} \quad A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}, \\ \overline{f} &:= \sup f(A, b, c) \quad \text{subject to} \quad A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}. \end{aligned}$$

These bounds are easily computable by two linear programs, as stated it the following theorem. For the proof see e.g. [6].

Theorem 1. *We have*

$$\underline{f} = \inf \underline{c}^T x \quad \textit{subject to} \quad \overline{A} x \geq \underline{b}, \ x \geq 0. \tag{2}$$

Let $\underline{f}$ *be finite or let the right-hand side of the equation* (3) *be positively infinite. Then*

$$\overline{f} = \sup \overline{b}^T y \quad \textit{subject to} \quad \underline{A}^T y \leq \overline{c}, \ y \geq 0. \tag{3}$$

The formula (2) holds without any assumption. To use the formula for the upper bound we must be sure that there is no duality gap between primal and dual linear programs (i.e., at least one of them to be feasible). Therefore we have the assumption in Theorem 1. If this assumption is not satisfied, we can process according to [6]: If $\underline{f} = \infty$ then clearly also $\overline{f} = \infty$. If $\underline{f} = -\infty$ then check feasibility of the linear system of inequalities

$$\underline{A} x \geq \overline{b}, \ x \geq 0.$$

If it is feasible then (3) is true, otherwise $\overline{f} = \infty$.

2 Computing the tolerances

We use the previous results for solving the inverse problem: We are given a linear program and the bounds for the optimal value. The aim is to compute intervals for all the interested quantities such that all of them can simultaneously and independently vary inside their intervals while the corresponding optimal value does not exceed the prescribed bounds.

Formally, consider the linear program

$$\min c_c^T x \quad \text{subject to} \quad A_c x \geq b_c, x \geq 0,$$

where A_c, b_c and c_c are known and fixed. By f^* denote its optimal value. Let A_Δ, be a nonnegative matrix representing scale of demanded perturbations. Its entries $(a_\Delta)_{ij}$ are usually set to ones (for absolute tolerances) or to $|(a_c)_{ij}|$ (for relative tolerances). If the quantity $(a_c)_{ij}$ is out of focus then we put $(a_\Delta)_{ij} = 0$. In the same manner introduce nonnegative vectors b_Δ and c_Δ.

Let "f'' and $\overline{f}$ be lower and upper bound of the optimal value, respectively, and suppose that "$f'' < f^* < \overline{f}$. We are seeking for a maximal $\delta > 0$ such that optimal value to (1) lies within the interval ["f'', $\overline{f}$] for all $A \in \boldsymbol{A}_\delta$, $b \in \boldsymbol{b}_\delta$ and $c \in \boldsymbol{c}_\delta$, where

$$\begin{aligned}\boldsymbol{A}_\delta &:= [A_c - \delta \cdot A_\Delta, A_c + \delta \cdot A_\Delta],\\ \boldsymbol{b}_\delta &:= [b_c - \delta \cdot b_\Delta, b_c + \delta \cdot b_\Delta],\\ \boldsymbol{c}_\delta &:= [c_c - \delta \cdot c_\Delta, c_c + \delta \cdot c_\Delta].\end{aligned}$$

Theorem 2. *Let*

$$\delta^1 := \inf \delta \text{ subject to } -c_c^T x + \delta \cdot c_\Delta^T x + \text{“}f'' \geq 0, \tag{4}$$
$$A_c x - b_c + \delta \cdot \big(A_\Delta x + b_\Delta\big) \geq 0,\ x \geq 0,$$
$$\delta^2 := \inf \delta \text{ subject to } b_c^T y + \delta \cdot b_\Delta^T y - \overline{f} \geq 0, \tag{5}$$
$$-A_c^T y + c_c + \delta \cdot \big(A_\Delta^T y + c_\Delta\big) \geq 0,\ y \geq 0,$$

Denote $\delta^* := \min(\delta^1, \delta^2) - \varepsilon$ with an arbitrarily small $\varepsilon > 0$. If the linear system

$$\big(A_c - \delta^* \cdot A_\Delta\big)x \geq b_c + \delta^* \cdot b_\Delta \tag{6}$$

is feasible then $f(A, b, c) \in$ ["f'', $\overline{f}$] for all $A \in \boldsymbol{A}_{\delta^*}$, $b \in \boldsymbol{b}_{\delta^*}$ and $c \in \boldsymbol{c}_{\delta^*}$.

Proof. The value of δ^1 is defined such that the optimal value function $f(A, b, c)$ does not get over the lower bound "f'' while perturbing the input data within $[0, \delta^1)$ tolerances. Let $\delta > 0$. According to Theorem 1, the minimal optimal value of the linear programs (1) over $A \in \boldsymbol{A}_\delta$, $b \in \boldsymbol{b}_\delta$ and $c \in \boldsymbol{c}_\delta$ is achieved for

$$\inf\ (c_c - \delta \cdot c_\Delta)^T x \ \text{ subject to } \ (A_c + \delta \cdot A_\Delta)x \geq b_c - \delta \cdot b_\Delta,\ x \geq 0. \tag{7}$$

We are seeking for a maximal $\delta > 0$ such that the linear program (7) does not exceed the lower bound "f''. This is not an easy problem in general, so we transform this problem to the another yielding the same or smaller value: We compute a minimal $\delta > 0$ such that the linear program (7) has a feasible point with the objective value being at most "f'' (since with the increase of δ the feasible set is expanding and the optimal value function is nonincreasing). Thus δ^1 can be obtained by solving the optimization problem

$$\delta^1 = \inf \delta \ \text{ subject to } \ c_c^T x - \delta \cdot c_\Delta^T x \leq \text{“}f'',\ \big(A_c + \delta \cdot A_\Delta\big)x \geq b_c - \delta \cdot b_\Delta,\ x \geq 0.$$

Nevertheless, the optimal value of (7) with $\delta = \delta^1$ can exceed the lower bound "f''. Therefore, we must subtract from δ^1 an arbitrarily small $\varepsilon > 0$.

The second part of the proof concerning δ^2 and the upper bound $\overline{f}$ is analogous and the condition (6) ensures that the formula (3) is applicable. Let $\delta > 0$. By (3), the maximal optimal value of the linear programs (1) over $A \in \boldsymbol{A}_\delta$, $b \in \boldsymbol{b}_\delta$ and $c \in \boldsymbol{c}_\delta$ is achieved for

$$\sup\ (b_c + \delta \cdot b_\Delta)^T y \ \text{ subject to } \ (A_c - \delta \cdot A_\Delta)^T y \leq c_c + \delta \cdot c_\Delta,\ y \geq 0, \tag{8}$$

and we want to maximize δ such that this linear program does not exceed the upper bound $\overline{f}$. Again, we transform a problem and compute a minimal $\delta > 0$ such that the linear program (8) has a feasible point with the objective value being at least $\overline{f}$. Thus δ^2 will be achieved by solving the optimization problem

$$\delta^2 = \inf \delta \;\text{ subject to }\; b_c^T y + \delta \cdot b_\Delta^T y \geq \overline{f},\; \left(A_c - \delta \cdot A_\Delta\right)^T y \leq c_c + \delta \cdot c_\Delta,\; y \geq 0.$$

□ □

Theorem 2 gives the formulae for computing the demanded tolerances. The value of resulting δ^* is not the best possible in general, but the cases when δ^* is not optimal are very rare.

Note that in the case when (6) is not satisfied, we can easily decrease δ^* such that the inequality system would be satisfied. The procedure is quite similar to the one proposed in the previous proof.

Note also that the optimization problems (4)–(5) belong to the class of so called generalized linear fractional programs. These programs have the form of

$$\sup \left(\inf_i \frac{P_{i\bullet} x}{Q_{i\bullet} x} \right) \;\text{ subject to }\; Qx > 0,\; Rx \geq r,$$

(where $P_{i\bullet}$ and $Q_{i\bullet}$ denotes i-th row of P and Q, respectively), or,

$$\sup \alpha \;\text{ subject to }\; Px - \alpha \cdot Qx \geq 0,\; Qx \geq 0,\; Rx \geq r,$$

and they are solvable in polynomial time using an interior point method [4], [12]. In the cases (4) and (5), the nonnegativity condition $Qx \geq 0$ is always satisfied, and therefore we omit it.

3 Application in portfolio selection problem

Interval analysis in portfolio selection problem was used e.g. in [7], [9], [10], [13]. Our approach is different; we exhibit results from the previous section to compute tolerances for the problem quantities.

Consider the following portfolio problem. We have J possible investments for T time periods and r_{jt}, $j = 1, \ldots, J$, $t = 1, \ldots, T$, stands for return on investment j in time period t. Estimated reward on investment j using historical means is defined as

$$R_j := \frac{1}{T} \sum_{t=1}^{T} r_{jt}.$$

In order to get a linear programming problem we measure risk of investment j by sum of absolute values instead of the historical variances:

$$\frac{1}{T} \sum_{t=1}^{T} |r_{jt} - R_j|.$$

Let μ be a risk aversion parameter (upper bound for risk) given by a user, and the variable x_j, $j = 1, \ldots, J$, denotes a fraction of portfolio to invest in j. Then the maximal allowed risk is expressed by the constraint

$$\frac{1}{T}\sum_{t=1}^{T}\Big|\sum_{j=1}^{J}(r_{jt}-R_j)x_j\Big| \le \mu,$$

or, by converting to linear inequality system

$$-y_j \le \sum_{j=1}^{J}(r_{jt}-R_j)x_j \le y_t,\ \forall t=1,\dots,T,\quad \frac{1}{T}\sum_{t=1}^{T} y_t \le \mu.$$

The portfolio selection problem takes the form

$$\begin{aligned} \max\ & \sum_{j=1}^{J} R_j x_j \\ \text{subject to}\ & -y_j \le \sum_{j=1}^{J}(r_{jt}-R_j)x_j \le y_t, \quad \forall t=1,\dots,T, \\ & \sum_{j=1}^{J} x_j = 1,\ \frac{1}{T}\sum_{t=1}^{T} y_t \le \mu, \\ & x_j \ge 0, \quad \forall j=1,\dots,J. \end{aligned}$$

Computing tolerances is demonstrated by the following example.

Consider a portfolio selection problem with $J = 4$ investments and $T = 5$ time periods. The risk aversion parameter is set as $\mu := 10$. The returns are displayed below:

time period t	reward on investment			
	1	2	3	4
1	11	20	9	10
2	13	25	11	13
3	10	17	12	11
4	12	21	11	13
5	12	19	13	14

The optimal solution is

$$x^* = (0,\ 0.1413,\ 0.8587,\ 0)^T,$$
$$y^* = (10.3891,\ 9.9043,\ 9.9587,\ 10.0174,\ 9.7304)^T,$$

and the corresponding optimal return is 12.5.

Now, we compute tolerances for the given rewards r_{jt} such that the optimal return does not exceed the upper bound $\overline{f} := 20$ and the lower bound "f" $:= 6$. Notice that the portfolio selection problem has an opposite direction of optimization in comparison with (1) (maximization instead of minimization).

First, we compute tolerances only for one quantity, say r_{21}. This quantity appears in the objective function and also several times in the constraints, and the traditional sensitivity analysis techniques are not applicable. Our approach works, though the results are not optimal just because of the multiple appearance of r_{21} causing dependences between the coefficients. We set the radii of intervals as $(r_\Delta)_{21} = 1$ and $(r_\Delta)_{jt} = 0$ otherwise. By solving the optimization problems (4) and (5) we get the results

$$\delta^1 = \infty, \quad \delta^2 = 14.8069,$$

and the condition (6) is fulfilled. Hence, the upper bound $\overline{f}$ can be possibly exceeded by perturbing r_{21} at least by $\delta^2 = 14.8069$, while the lower bound will never be achieved or exceeded.

Now, we take into account the quantities r_{2t}, $t = 1, \ldots, T$. Define the radii $(r_\Delta)_{2t} = 1$, $t = 1, \ldots, T$, and $(r_\Delta)_{jt} = 0$, $j \neq 2$, $t = 1, \ldots, T$. By solving (4) and (5) we obtain

$$\delta^1 = \infty, \quad \delta^2 = 2.9614,$$

and the condition (6) is again fulfilled.

In the last case we take into account all the quantities r_{jt}, $j = 1, \ldots, J$, $t = 1, \ldots, T$. The radii are set accordingly as $(r_\Delta)_{jt} = 1$, $j = 1, \ldots, J$, $t = 1, \ldots, T$. Naturally, computations yield much smaller values

$$\delta^1 = 0.04545, \quad \delta^2 = 0.1575.$$

The resulting tolerance is $\delta^* = 0.04545 - \varepsilon$ with an arbitrarily small $\varepsilon > 0$, and the condition (6) is fulfilled. Therefore, as long as all the rewards r_{jt} vary within intervals $[r_{jt} - \delta^*, r_{jt} + \delta^*]$, the optimal return will never get outside the interval $[6, 20]$.

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