Tolerances in portfolio selection via interval linear programming

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Introduction

Definition

- An interval matrix

$$A^I = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\},$$

- Midpoint and radius of A^I

$$A_c \equiv \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta \equiv \frac{1}{2}(\overline{A} - \underline{A}).$$

Definition

Interval linear program is a family

$$f(A, b, c) \equiv \min c^T x \;\text{ subject to }\; Ax \geq b, x \geq 0,$$

where $A \in A^I$, $b \in b^I$, $c \in c^I$.

Introduction

Definition

Lower and upper bounds of the optimal value

$$\underline{f} := \inf f(A, b, c) \ \text{ subject to } \ A \in A^I, b \in b^I, c \in c^I,$$
$$\overline{f} := \sup f(A, b, c) \ \text{ subject to } \ A \in A^I, b \in b^I, c \in c^I.$$

Theorem

We have

$$\underline{f} = \inf \underline{c}^T x \ \textit{ subject to } \ \overline{A}x \geq \underline{b}, \ x \geq 0.$$

If $\underline{A}x \geq \overline{b}, \ x \geq 0$ *is feasible then*

$$\overline{f} = \sup \overline{b}^T y \ \textit{ subject to } \ \underline{A}^T y \leq \overline{c}, \ y \geq 0.$$

Problem statement

Consider

$$\min c_c^T x \quad \text{subject to} \quad A_c x \geq b_c, x \geq 0,$$

a let f^* be its optimal value. Given:

- bounds $\overline{f} < f^* < \overline{f}$;
- perturbation rates $A_\Delta, b_\Delta, c_\Delta \geq 0$.

Our goal

Find maximal $\delta > 0$ such that

$$\underline{f} \leq f(A, b, c) \leq \overline{f}$$

for all $A \in A^I_\delta$, $b \in b^I_\delta$ and $c \in c^I_\delta$, where

$$A^I_\delta := [A_c - \delta \cdot A_\Delta, A_c + \delta \cdot A_\Delta], \quad b^I_\delta := [b_c - \delta \cdot b_\Delta, b_c + \delta \cdot b_\Delta],$$
$$c^I_\delta := [c_c - \delta \cdot c_\Delta, c_c + \delta \cdot c_\Delta].$$

Main result

Theorem

Let

$$\delta^1 := \inf \delta \textit{ subject to } -c_c^T x + \delta \cdot c_\Delta^T x + \underline{f} \geq 0,$$
$$A_c x - b_c + \delta \cdot \left(A_\Delta x + b_\Delta\right) \geq 0,\ x \geq 0,$$
$$\delta^2 := \inf \delta \textit{ subject to } b_c^T y + \delta \cdot b_\Delta^T y - \overline{f} \geq 0,$$
$$-A_c^T y + c_c + \delta \cdot \left(A_\Delta^T y + c_\Delta\right) \geq 0,\ y \geq 0,$$

Let $0 \leq \delta^* < \min(\delta^1, \delta^2)$. *If the linear system*

$$\left(A_c - \delta^* \cdot A_\Delta\right) x \geq b_c + \delta^* \cdot b_\Delta$$

is feasible then $f(A, b, c) \in [\underline{f}, \overline{f}]$ *for all* $A \in A^I_{\delta*}$, $b \in b^I_{\delta*}$ *and* $c \in c^I_{\delta*}$.

Main result

Remark

- The optimization problems have the form of generalized linear fractional programs. These programs have the form of

$$\sup \left(\inf_i \frac{P_{i\cdot}x}{Q_{i\cdot}x} \right) \text{ subject to } Qx > 0,\ Rx \geq r,$$

(where $P_{i\cdot}$ and $Q_{i\cdot}$ denotes i-th row of P and Q, respectively), or,

$$\sup \alpha \text{ subject to } Px - \alpha \cdot Qx \geq 0,\ Qx \geq 0,\ Rx \geq r,$$

They are solvable in polynomial time using an interior point method.

- Generally, δ^* is not the best possible, but mostly it is.

Application: portfolio selection

Given:

- J possible investments;
- T time periods;
- r_{jt}, return on investment j in time period t;
- μ, risk aversion parameter (upper bound for risk).

Then:

- Estimated reward on investment j: $R_j := \frac{1}{T}\sum_{t=1}^{T} r_{jt}$;
- Risk measure of investment j: $\frac{1}{T}\sum_{t=1}^{T} |r_{jt} - R_j|$;
- Maximal allowed risk: $\frac{1}{T}\sum_{t=1}^{T} \left| \sum_{j=1}^{J} (r_{jt} - R_j)x_j \right| \leq \mu.$

Application: portfolio selection

Portfolio selection problem formulation

$$\max \sum_{j=1}^{J} R_j x_j$$

$$\text{subject to } -y_j \leq \sum_{j=1}^{J} (r_{jt} - R_j) x_j \leq y_t, \quad \forall t = 1, \dots, T,$$

$$\sum_{j=1}^{J} x_j = 1, \; \frac{1}{T} \sum_{t=1}^{T} y_t \leq \mu,$$

$$x_j \geq 0, \quad \forall j = 1, \dots, J,$$

where

$$R_j := \frac{1}{T} \sum_{t=1}^{T} r_{jt}.$$

Example

Example

$J = 4$ investments, $T = 5$ time periods, $\mu = 10$ risk aversion parameter.
The returns:

time period t	reward on investment			
	1	2	3	4
1	11	20	9	10
2	13	25	11	13
3	10	17	12	11
4	12	21	11	13
5	12	19	13	14

The optimal return is 12.5.

Example

Tolerances for optimal return being within [6, 20]:

1. Put $(r_\Delta)_{21} = 1$, and $(r_\Delta)_{jt} = 0$ otherwise. We get

$$\delta^1 = \infty, \quad \delta^2 = 14.8069.$$

2. Put $(r_\Delta)_{2t} = 1$, $t = 1, \ldots, T$, and $(r_\Delta)_{jt} = 0$, $j \neq 2$, $t = 1, \ldots, T$. We get

$$\delta^1 = \infty, \quad \delta^2 = 2.9614.$$

3. Put $(r_\Delta)_{jt} = 1$, $j = 1, \ldots, J$, $t = 1, \ldots, T$. We get

$$\delta^1 = 0.04545, \quad \delta^2 = 0.1575.$$

Last slide

The End.