Optimal value bounds in nonlinear programming with interval data

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Introduction

Motivation: data uncertainty.

Definition

- An interval matrix

$$A^I = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\},$$

- Midpoint and radius of A^I

$$A_c \equiv \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta \equiv \frac{1}{2}(\overline{A} - \underline{A}).$$

Consider the interval nonlinear program

$$f(A, c) \equiv \inf f_c(x) \text{ subject to } F_A(x) \le 0,$$

where $A \in A^I$, $c \in c^I$.

Introduction

Lower and upper bounds of the optimal value

$$\underline{f} \equiv \inf f(A,c) \ \text{ subject to } \ A \in A^I, c \in c^I,$$
$$\overline{f} \equiv \sup f(A,c) \ \text{ subject to } \ A \in A^I, c \in c^I.$$

Dual problem

$$g(A,c) \equiv \sup g_{A,c}(y) \ \text{ subject to } \ G_{A,c}(y) \le 0,$$

Suppose weak duality

$$f(A,c) \ge g(A,c) \ \ \forall c \in c^I, A \in A^I.$$

Zero duality gap

$$f(A,c) = g(A,c) \ \ \forall c \in c^I, A \in A^I.$$

Introduction

Define functions

$$f(x) \equiv \inf f_c(x) \;\; \text{subject to} \;\; c \in c^I,$$
$$g(y) \equiv \sup g_{A,c}(y) \;\; \text{subject to} \;\; A \in A^I, c \in c^I,$$

and solution sets

$$M \equiv \{x \in \mathbb{R}^n \mid F_A(x) \leq 0, A \in A^I\},$$
$$N \equiv \{y \in \mathbb{R}^k \mid G_{A,c}(y) \leq 0, A \in A^I, c \in c^I\}.$$

Main theorem

Theorem

We have

$$\underline{f} = \inf f(x) \quad \textit{subject to} \quad x \in M.$$

If the zero duality gap is guaranteed, then

$$\overline{f} \leq \sup g(y) \quad \textit{subject to} \quad y \in N.$$

If the functions $G_{A,c}(y)$ and $g_{A,c}(y)$ have no interval parameter in common, then

$$\overline{f} \geq \sup g(y) \quad \textit{subject to} \quad y \in N.$$

Applications

Convex quadratic programming:

- inventory management
- economics (portfolio selection)
- engineering design, molecular study

Posynomial geometric programming:

- inventory and project management
- power control in communications systems
- engineering design (integrated circuits and gate sizing, truss construction)

Convex quadratic programming

Consider a family

$$\min x^T Cx + d^T x \;\text{ subject to }\; Ax \le b, x \ge 0,$$

where $A \in A^I$, $C \in C^I$, $b \in b^I$, $d \in d^I$.

Suppose that C is a positive semidefinite $\forall C \in C^I$.

Dorn dual

$$\max -x^T Cx - b^T u \;\text{ subject to }\; 2Cx + A^T u + d \ge 0, u \ge 0.$$

Lower bound

$$\underline{f} = \inf x^T \underline{C}x + \underline{d}^T x \;\text{ subject to }\; \underline{A}x \le \overline{b},\; x \ge 0.$$

Convex quadratic programming

Upper bound (achieved for $C = \overline{C}$)

$$\overline{f} \geq \sup -x^T\overline{C}x - \underline{b}^T u \ \text{ subject to } \ 2\overline{C}x + \overline{A}^T u + \overline{d} \geq 0, u \geq 0.$$

If duality gap is zero

$$\overline{f} = \sup -x^T\overline{C}x - \underline{b}^T u \ \text{ subject to } \ 2\overline{C}x + \overline{A}^T u + \overline{d} \geq 0, u \geq 0.$$

Sufficient conditions for zero duality gap:

- Feasibility of $\overline{A}x \leq \underline{b},\ x \geq 0$.
- Feasibility of $2\underline{C}x^1 - 2\overline{C}x^2 + \underline{A}^T u + \underline{d} \geq 0,\ u, x^1, x^2 \geq 0$.
- Positive definiteness for all $C \in \mathcal{C}$.

Convex quadratic programming

Example (modified Liu & Wang, 2007)

Let

$$C = \begin{pmatrix} [2,3] & -1 \\ -1 & 2 \end{pmatrix}, \; d = \begin{pmatrix} [-5,-3] \\ [1,2] \end{pmatrix}, \; A = \begin{pmatrix} [1,2] & 1 \\ [2,3] & [-1,-0.5] \end{pmatrix}, \; b = \begin{pmatrix} [2,4] \\ [3,4] \end{pmatrix}.$$

Lower bound

$$\underline{f} = \inf x^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x + \begin{pmatrix} -5 \\ 1 \end{pmatrix} x \;\text{ subject to }\; \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} x \le \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \; x \ge 0$$

$$= -3.5$$

Duality gap is zero, thus

$$\overline{f} = \sup -x^T \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} x - \begin{pmatrix} 2 \\ 3 \end{pmatrix} u$$

$$\text{subject to }\; 2 \cdot \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} x + \begin{pmatrix} 2 & 3 \\ 3 & -0.5 \end{pmatrix} u + \begin{pmatrix} -3 \\ 2 \end{pmatrix} \ge 0, \; u \ge 0$$

$$\simeq -0.4821$$

Optimal value range is $[-3.5, -0.4821]$.

Posynomial geometric programming

Consider

$$\inf \sum_{i \in I_0} c_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

subject to

$$\sum_{i \in I_k} c_i \prod_{j=1}^{n} x_j^{a_{ij}} \leq 1, \quad k = 1, \ldots, m,$$

$$x_j > 0, \quad j = 1, \ldots, n.$$

where $I_0 = \{1, \ldots, p_0\}$, $I_1 = \{p_0 + 1, \ldots, p_1\}$, $\ldots$, $I_m = \{p_{m-1} + 1, \ldots, p\}$, and $c_i > 0$, $i = 1, \ldots, p$.

Posynomial geometric programming

Dual problem

$$\sup \left(\prod_{i=1}^{p} \left(\frac{c_i}{y_i} \right)^{y_i} \right) \left(\prod_{k=1}^{m} z_k^{z_k} \right)$$

subject to

$$\begin{aligned} \sum_{i \in I_0} y_i &= 1, \\ \sum_{i \in I_k} y_i &= z_k, \quad k = 1, \ldots, m, \\ \sum_{i=1}^{p} a_{ij} y_i &= 0, \quad j = 1, \ldots, n, \\ y_i, z_k &\geq 0, \quad i = 1, \ldots, p, \;\; k = 1, \ldots, m. \end{aligned}$$

Duality gap is zero if the primal problem has an interior point.

Posynomial geometric programming

Suppose $c_i \in c^I{}_i$ and $a_{ij} \in a^I{}_{ij}$ $\forall i = 1, \ldots, p,\; j = 1, \ldots, n$.

Proposition (sufficient condition for zero duality gap)

If the system

$$\sum_{i \in I_k} \overline{c}_i \prod_{j=1}^{n} y_j^{\overline{a}_{ij}} z_j^{-\underline{a}_{ij}} < 1, \quad k = 1, \ldots, m,$$

$$y_j, z_j \geq 1, \quad j = 1, \ldots, n.$$

has a solution y^*, z^*, then the vector x^* defined as $x_i^* \equiv \frac{y_i^*}{z_i^*}$ is an interior point of the primal problem for all $c_i \in c^I{}_i$ and $a_{ij} \in a^I{}_{ij}$.

Remark

- Easy to convert to posynomial geometric program
- Open: Is it also necessary condition for x^* being interior?

Posynomial geometric programming

Lower bound (attained at $c_i = \underline{c}_i$)

$$\underline{f} = \inf f(x) \;\text{ subject to }\; x \in M.$$

We have $a_{ij} = \underline{a}_{ij}$ if $x_j \geq 1$ and $a_{ij} = \overline{a}_{ij}$ if $x_j < 1$. Hence

$$f(x) = \sum_{i \in I_0} \underline{c}_i \prod_{j=1}^{n} x_j^{(a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j))}.$$

M is described by

$$\sum_{i \in I_k} \underline{c}_i \prod_{j=1}^{n} x_j^{(a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j))} \leq 1, \quad k = 1, \ldots, m,$$

$$x_j > 0, \quad j = 1, \ldots, n.$$

Posynomial geometric programming

Upper bound (attained at $c_i = \overline{c}_i$)

$$\overline{f} = \sup g(y,z) \;\text{ subject to }\; (y,z) \in N,$$

provided the duality gap is zero. Here,

$$g(y,z) = \left(\prod_{i=1}^{p} \left(\frac{\overline{c}_i}{y_i} \right)^{y_i} \right) \left(\prod_{k=1}^{m} z_k^{z_k} \right),$$

and

$$\sum_{i \in I_0} y_i = 1,$$
$$\sum_{i \in I_k} y_i = z_k, \quad k = 1, \ldots, m,$$
$$\sum_{i=1}^{p} \overline{a}_{ij} y_i, \geq 0,\; \sum_{i=1}^{p} \underline{a}_{ij} y_i \leq 0, \quad j = 1, \ldots, n,$$
$$y_i, z_k \geq 0, \quad i = 1, \ldots, p,\; k = 1, \ldots, m.$$

The End.