Linear interval systems with a specific dependence structure

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Abstract

This is a contribution to solvability of linear interval equations and inequalities. In interval analysis we usually suppose that values from different intervals are mutually independent. This assumption can be sometimes too restrictive. In this article we derive extensions of Oettli–Prager theorem and Gerlach theorem for the case there is a simple dependence structure between coefficients of an interval system. The dependence is given by equality of two submatrices of the constraint matrix.

Keywords: *interval systems, interval matrix, weak solution.*

1 Introduction

Coefficients and right-hand sides of systems of linear equalities and inequalities are rarely known exactly. In interval analysis we suppose that these values vary in some real intervals independently. But in practical applications (for instance electrical circuit problem [5, 6]) they are sometimes related. General case of parametric dependences has been considered e.g. in [7, 9], where various algorithms for finding inner and outer solutions were proposed. Linear interval systems with more specific dependencies were studied e.g. in [1, 2]. There were derived basic characteristics (shape, enclosures, etc.), especially for cases where the constraint matrix is supposed to be symmetric or skew-symmetric. But any explicit condition such as Oettli–Prager theorem [8] (for linear interval equations) or Gerlach Theorem [4] (for linear interval inequalities) has never appeared.

In this paper we focus on weak solvability of linear interval systems with a simple dependence structure and derive explicit (generally nonlinear) conditions for such a solvability.

Let us introduce some notation. The i-th row of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}_{i,\cdot}$, the j-th column by $\mathbf{A}_{\cdot,j}$. The vector $\mathbf{e} = (1, \ldots, 1)^T$ is the vector of all ones. An interval matrix is defined as

$$\mathbf{A}^I = [\underline{\mathbf{A}}, \overline{\mathbf{A}}] = \{\mathbf{A} \in \mathbb{R}^{m\times n} \mid \underline{\mathbf{A}} \leq \mathbf{A} \leq \overline{\mathbf{A}}\},$$

where $\underline{\mathbf{A}} \le \overline{\mathbf{A}}$ are fixed matrices. By

$$\mathbf{A}^c \equiv \frac{1}{2}(\overline{\mathbf{A}} + \underline{\mathbf{A}}), \quad \mathbf{A}^\Delta \equiv \frac{1}{2}(\overline{\mathbf{A}} - \underline{\mathbf{A}})$$

we denote the midpoint and radius of $\mathbf{A}^I$, respectively. The interval matrix addition and subtraction is defined as follows

$$\begin{aligned} \mathbf{A}^I + \mathbf{B}^I &= [\underline{\mathbf{A}} + \underline{\mathbf{B}}, \overline{\mathbf{A}} + \overline{\mathbf{B}}], \\ \mathbf{A}^I - \mathbf{B}^I &= [\underline{\mathbf{A}} - \overline{\mathbf{B}}, \overline{\mathbf{A}} - \underline{\mathbf{B}}]. \end{aligned}$$

A vector $\mathbf{x} \in \mathbb{R}^n$ is called a *weak solution* of a linear interval system $\mathbf{A}^I\mathbf{x} = \mathbf{b}^I$, if $\mathbf{A}\mathbf{x} = \mathbf{b}$ holds for some $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{b} \in \mathbf{b}^I$. Analogously we define a term weak solution for other types of interval systems (cf. [3]).

2 Generalization of Oettli–Prager theorem

In this section we generalize the Oettli–Prager [8] characterization of weak solutions of linear interval equations to the case where there is a specific dependence between some coefficients of the constraint matrix.

Lemma 1. *Given $s_1,\ s_2,\ p_i,\ q_i \in \mathbb{R},\ i = 1,\dots,n$. Let us denote the function $f(u_1, u_2) \equiv s_1u_1 + s_2u_2 + \sum_{i=1}^n |p_iu_1 + q_iu_2|$. Then the problem*

$$\min\{f(u_1, u_2);\ (u_1, u_2) \in \mathbb{R}^2\} \tag{1}$$

has an optimal solution (equal to zero) if and only if

$$\begin{aligned} \sum_{i=1}^n |q_i| &\ge |s_2|, \\ \sum_{i=1}^n |q_kp_i - q_ip_k| &\ge |q_ks_1 - p_ks_2| \quad \forall k = 1,\dots,n \end{aligned}$$

holds.

Proof. The objective function $f(u_1, u_2)$ is positive homogeneous and hence the problem (1) has an optimal solution iff $f(u_1, u_2) \ge 0$ holds for $u_1 = \pm 1,\ u_2 \in \mathbb{R}$ and for $u_1 = 0,\ u_2 = \pm 1$. Let us consider the following cases:

(i) Let $u_1 = 1$. Then the function $f(1, u_2) = s_1 + s_2u_2 + \sum_{i=1}^n |p_i + q_iu_2|$ of one parameter represents a broken line. It is sufficient to check nonnegativity of this function in the breaks and nonnegativity of the limits in $\pm\infty$. The breaks are $-\frac{p_k}{q_k}$, $q_k \ne 0,\ k = 1,\dots,n$. Hence we derive

$$\forall k = 1,\dots,n,\ q_k \ne 0:\ s_1 - \frac{p_k}{q_k}s_2 + \sum_{i=1}^n \left|p_i - \frac{p_k}{q_k}q_i\right| \ge 0. \tag{2}$$

To be $\lim_{u_2\to\infty} f(1, u_2) \ge 0$, it must the inequality $\sum_{i=1}^n |q_i| \ge -s_2$ hold and to be $\lim_{u_2\to-\infty} f(1, u_2) \ge 0$, it must $\sum_{i=1}^n |q_i| \ge s_2$ hold. We obtain next condition

$$\sum_{i=1}^n |q_i| \ge |s_2|. \tag{3}$$

(ii) Let $u_1 = -1$. Then analogously as in first paragraph we obtain for the function $f(-1, u_2) = -s_1 + s_2 u_2 + \sum_{i=1}^n |-p_i + q_i u_2|$ the condition

$$\forall k = 1, \ldots, n,\ q_k \neq 0: \ -s_1 + \frac{p_k}{q_k} s_2 + \sum_{i=1}^n \left| -p_i + \frac{p_k}{q_k} q_i \right| \geq 0, \tag{4}$$

$$\sum_{i=1}^n |q_i| \geq |s_2|. \tag{5}$$

All the conditions (2), (4) can we written in one

$$\forall k = 1, \ldots, n: \ \sum_{i=1}^n |p_i q_k - p_k q_i| \geq |s_1 q_k - p_k s_2|. \tag{6}$$

The assumption $q_k \neq 0$ is not necessary, for in the case $q_k = 0$ the inequality (6) is included in (3).

(iii). Let $u_1 = 0$. Then the condition $f(0, \pm 1) \geq 0$ is included in the condition (3). □

Theorem 1. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$, $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$, $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$. *Then for certain* $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B} \in \mathbf{B}^I$, $\mathbf{C} \in \mathbf{C}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$ *vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ *form a solution of the system*

$$\mathbf{Ax} + \mathbf{Bz} = \mathbf{b}, \tag{7}$$

$$\mathbf{Ay} + \mathbf{Cz} = \mathbf{c} \tag{8}$$

if and only if they satisfy the following system of inequalities

$$\mathbf{A}^\Delta|\mathbf{x}| + \mathbf{B}^\Delta|\mathbf{z}| + \mathbf{b}^\Delta \geq |\mathbf{r}_1|, \tag{9}$$

$$\mathbf{A}^\Delta|\mathbf{y}| + \mathbf{C}^\Delta|\mathbf{z}| + \mathbf{c}^\Delta \geq |\mathbf{r}_2|, \tag{10}$$

$$\mathbf{B}^\Delta|\mathbf{z}||\mathbf{y}|^T + \mathbf{C}^\Delta|\mathbf{z}||\mathbf{x}|^T + \mathbf{b}^\Delta|\mathbf{y}|^T + \mathbf{c}^\Delta|\mathbf{x}|^T + \mathbf{A}^\Delta|\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T| \geq |\mathbf{r}_1\mathbf{y}^T - \mathbf{r}_2\mathbf{x}^T|, \tag{11}$$

where $\mathbf{r}_1 \equiv -\mathbf{A}^c\mathbf{x} - \mathbf{B}^c\mathbf{z} + \mathbf{b}^c$, $\mathbf{r}_2 \equiv -\mathbf{A}^c\mathbf{y} - \mathbf{C}^c\mathbf{z} + \mathbf{c}^c$.

Proof. Denote $\mathbf{a}^I \equiv \mathbf{A}^I_{l,\cdot}$, $\mathbf{b}^I \equiv \mathbf{B}^I_{l,\cdot}$, $\mathbf{c}^I \equiv \mathbf{C}^I_{l,\cdot}$, $\beta^I \equiv \mathbf{b}^I_l$, $\gamma^I \equiv \mathbf{c}^I_l$. Consider the l-th equations in systems (7)–(8) and denote them by

$$\mathbf{ax} + \mathbf{bz} = \beta, \ \mathbf{ay} + \mathbf{cz} = \gamma, \tag{12}$$

where $\mathbf{a} \in \mathbf{a}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$, $\beta \in \beta^I$, $\gamma \in \gamma^I$. Suppose that the vector $\mathbf{a} \in \mathbf{a}^I$ in demand has the i-th component in the form $a_i \equiv a_i^c + \alpha_i a_i^\Delta$ for $\alpha_i \in \langle -1, 1\rangle$. The condition (12) holds iff for a certain $\boldsymbol{\alpha} \in \langle -1, 1\rangle^n$ relations

$$\mathbf{a}^c\mathbf{x} + \sum_{i=1}^n \alpha_i a_i^\Delta x_i + \mathbf{b}^c\mathbf{z} \in \langle \beta^c - \mathbf{b}^\Delta|\mathbf{z}| - \beta^\Delta, \beta^c + \mathbf{b}^\Delta|\mathbf{z}| + \beta^\Delta\rangle,$$

$$\mathbf{a}^c\mathbf{y} + \sum_{i=1}^n \alpha_i a_i^\Delta y_i + \mathbf{c}^c\mathbf{z} \in \langle \gamma^c - \mathbf{c}^\Delta|\mathbf{z}| - \gamma^\Delta, \gamma^c + \mathbf{c}^\Delta|\mathbf{z}| + \gamma^\Delta\rangle$$

hold. Equivalently, iff the following problem

$$\max\Big\{\mathbf{0}^T\boldsymbol{\alpha};\ -\sum_{i=1}^n \alpha_i a_i^\Delta x_i \le -r_1+\beta_1,\quad \sum_{i=1}^n \alpha_i a_i^\Delta x_i \le r_1+\beta_1,$$
$$-\sum_{i=1}^n \alpha_i a_i^\Delta y_i \le -r_2+\beta_2,\quad \sum_{i=1}^n \alpha_i a_i^\Delta y_i \le r_2+\beta_2,$$
$$\boldsymbol{\alpha}\le\mathbf{e},\quad -\boldsymbol{\alpha}\le\mathbf{e}\Big\}$$

has an optimal solution for $r_1 \equiv -\mathbf{a}^c\mathbf{x} - \mathbf{b}^c\mathbf{z} + \beta^c$, $r_2 \equiv -\mathbf{a}^c\mathbf{y} - \mathbf{c}^c\mathbf{z} + \gamma^c$, $\beta_1 \equiv \beta^\Delta + \mathbf{b}^\Delta|\mathbf{z}|$, $\beta_2 \equiv \gamma^\Delta + \mathbf{c}^\Delta|\mathbf{z}|$. From duality theory in linear programming this problem has an optimal solution iff the problem

$$\min\Big\{(-r_1+\beta_1)u_1 + (r_1+\beta_1)u_2 + (-r_2+\beta_2)u_3 + (r_2+\beta_2)u_4 + \sum_{i=1}^n (v_i+w_i);$$
$$-a_i^\Delta x_i u_1 + a_i^\Delta x_i u_2 - a_i^\Delta y_i u_3 + a_i^\Delta y_i u_4 + v_i - w_i = 0\ \forall i=1,\dots,n,$$
$$u_1,u_2,u_3,u_4,v_i,w_i \ge 0\ \forall i=1,\dots,n\Big\}$$

has an optimal solution. After substitution $\tilde u_1 \equiv u_2 - u_1$, $\tilde u_3 \equiv u_4 - u_3$ we can rewrite this problem as

$$\min\Big\{(r_1+\beta_1)\tilde u_1 + 2\beta_1 u_1 + (r_2+\beta_2)\tilde u_3 + 2\beta_2 u_3 + \sum_{i=1}^n (v_i+w_i);$$
$$a_i^\Delta x_i \tilde u_1 + a_i^\Delta y_i \tilde u_3 + v_i - w_i = 0\ \forall i=1,\dots,n,$$
$$u_1 \ge -\tilde u_1,\ u_3 \ge -\tilde u_3,\ u_1,u_3,v_i,w_i \ge 0\ \forall i=1,\dots,n\Big\}.$$

For optimal v_i, w_i, u_1, u_3 we have $u_1 = (-\tilde u_1)^+$, $u_3 = (-\tilde u_3)^+$, and $v_i + w_i = |a_i^\Delta x_i \tilde u_1 + a_i^\Delta y_i \tilde u_3|$ (since one of v_i, w_i is equal to zero). Hence the problem can be reformulated as

$$\min\Big\{(r_1+\beta_1)\tilde u_1 + 2\beta_1(-\tilde u_1)^+ + (r_2+\beta_2)\tilde u_3 + 2\beta_2(-\tilde u_3)^+$$
$$+\sum_{i=1}^n |a_i^\Delta x_i \tilde u_1 + a_i^\Delta y_i \tilde u_3|;\quad \tilde u_1,\tilde u_3\in\mathbb{R}\Big\}.$$

The positive part of real number p is equal to $p^+ = \frac{1}{2}(p+|p|)$ and the problem comes in the form

$$\min\Big\{r_1\tilde u_1 + \beta_1|\tilde u_1| + r_2\tilde u_3 + \beta_2|\tilde u_3| + \sum_{i=1}^n |a_i^\Delta x_i \tilde u_1 + a_i^\Delta y_i \tilde u_3|;\ \tilde u_1,\tilde u_3\in\mathbb{R}\Big\}. \quad (13)$$

Now we use Lemma 1 with u_1 replaced by $\tilde u_1$, u_2 replaced by $\tilde u_3$, n by $n+2$, and next $s_1 \equiv r_1$, $s_2 \equiv r_2$, $p_i \equiv a_i^\Delta x_i$ $(i = 1,\dots,n)$, $p_{n+1} \equiv \beta_1$, $p_{n+2} \equiv 0$, $q_i \equiv a_i^\Delta y_i$

$(i = 1, \ldots, n)$, $q_{n+1} \equiv 0$, $q_{n+2} \equiv \beta_2$. Hence the problem (13) has an optimum iff

$$\begin{aligned}
\sum_{i=1}^{n} a_i^{\Delta}|x_i| + \beta_1 &\geq |r_1|, \\
\sum_{i=1}^{n} a_i^{\Delta}|y_i| + \beta_2 &\geq |r_2|, \\
\beta_1|y_k| + \beta_2|x_k| + \sum_{i=1}^{n} a_i^{\Delta}|y_k x_i - x_k y_i| &\geq |y_k r_1 - x_k r_2| \ \forall k = 1, \ldots, n
\end{aligned}$$

holds, or, equivalently iff

$$\begin{aligned}
\mathbf{a}^{\Delta}|\mathbf{x}| + \mathbf{b}^{\Delta}|\mathbf{z}| + \beta^{\Delta} &\geq |r_1|, \\
\mathbf{a}^{\Delta}|\mathbf{y}| + \mathbf{c}^{\Delta}|\mathbf{z}| + \gamma^{\Delta} &\geq |r_2|, \\
\mathbf{b}^{\Delta}|\mathbf{z}||\mathbf{y}|^T + \beta^{\Delta}|\mathbf{y}|^T + \mathbf{c}^{\Delta}|\mathbf{z}||\mathbf{x}|^T + \gamma^{\Delta}|\mathbf{x}|^T + \mathbf{a}^{\Delta}|\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T| &\geq |r_1\mathbf{y}^T - r_2\mathbf{x}^T|
\end{aligned}$$

holds. These inequalities represents the l-th inequalities from systems (9)–(11), which proves the statement. □

In the case that $\mathbf{x} = \mathbf{y}$ we immediately have the following corollary.

Corollary 1. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$*,* $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$*,* $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$*. Then for certain* $\mathbf{A} \in \mathbf{A}^I$*,* $\mathbf{B} \in \mathbf{B}^I$*,* $\mathbf{C} \in \mathbf{C}^I$*,* $\mathbf{b} \in \mathbf{b}^I$*,* $\mathbf{c} \in \mathbf{c}^I$ *vectors* $\mathbf{x} \in \mathbb{R}^n$*,* $\mathbf{z} \in \mathbb{R}^h$ *form a solution of the system*

$$\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b}, \tag{14}$$

$$\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{z} = \mathbf{c} \tag{15}$$

if and only if they represent a weak solution of the linear interval system

$$\mathbf{A}^I\mathbf{x} + \mathbf{B}^I\mathbf{z} = \mathbf{b}^I, \tag{16}$$

$$\mathbf{A}^I\mathbf{x} + \mathbf{C}^I\mathbf{z} = \mathbf{c}^I, \tag{17}$$

$$\mathbf{B}^I\mathbf{z} - \mathbf{C}^I\mathbf{z} = \mathbf{b}^I - \mathbf{c}^I. \tag{18}$$

3 Generalization of Gerlach theorem

Now we generalize the Gerlach [4] characterization of weak solutions of linear interval inequalities to the case where there is a specific dependence between some coefficients of the constraint matrix.

Theorem 2. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$*,* $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$*,* $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$*.Then for certain* $\mathbf{A} \in \mathbf{A}^I$*,* $\mathbf{B} \in \mathbf{B}^I$*,* $\mathbf{C} \in \mathbf{C}^I$*,* $\mathbf{b} \in \mathbf{b}^I$*,* $\mathbf{c} \in \mathbf{c}^I$ *vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,* $\mathbf{z} \in \mathbb{R}^h$ *form a solution of the system*

$$\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} \leq \mathbf{b}, \tag{19}$$

$$\mathbf{A}\mathbf{y} + \mathbf{C}\mathbf{z} \leq \mathbf{c} \tag{20}$$

if and only if they satisfy the system of inequalities

$$
\begin{aligned}
\mathbf{A}^{\Delta}|\mathbf{x}| + \mathbf{B}^{\Delta}|\mathbf{z}| + \overline{\mathbf{b}} &\geq \mathbf{A}^c\mathbf{x} + \mathbf{B}^c\mathbf{z}, && (21)\\
\mathbf{A}^{\Delta}|\mathbf{y}| + \mathbf{C}^{\Delta}|\mathbf{z}| + \overline{\mathbf{c}} &\geq \mathbf{A}^c\mathbf{y} + \mathbf{C}^c\mathbf{z}, && (22)\\
\mathbf{r}_1|y_k| + \mathbf{r}_2|x_k| + \mathbf{A}^{\Delta}|y_k\mathbf{x} - x_k\mathbf{y}| &\geq \mathbf{0} \quad \forall k = 1,\dots,n : x_k y_k < 0, && (23)
\end{aligned}
$$

where $\mathbf{r}_1 \equiv \overline{\mathbf{b}} - \mathbf{A}^c\mathbf{x} - \mathbf{B}^c\mathbf{z} + \mathbf{B}^{\Delta}|\mathbf{z}|$, $\mathbf{r}_2 \equiv \overline{\mathbf{c}} - \mathbf{A}^c\mathbf{y} - \mathbf{C}^c\mathbf{z} + \mathbf{C}^{\Delta}|\mathbf{z}|$.

Proof. Denote $\mathbf{a}^I \equiv \mathbf{A}^I_{l,\cdot}$, $\mathbf{b}^I \equiv \mathbf{B}^I_{l,\cdot}$, $\mathbf{c}^I \equiv \mathbf{C}^I_{l,\cdot}$, $\beta^I \equiv \mathbf{b}^I_l$, $\gamma^I \equiv \mathbf{c}^I_l$. Let us consider the l-th inequalities in systems (19)–(20) and denote them by

$$\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{z} \leq \beta,\ \ \mathbf{a}\mathbf{y} + \mathbf{c}\mathbf{z} \leq \gamma, \tag{24}$$

where $\mathbf{a} \in \mathbf{a}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$, $\beta \in \beta^I$, $\gamma \in \gamma^I$. Let us consider the vector in demand $\mathbf{a} \in \mathbf{a}^I$ in the form with the i-th component $a_i \equiv a_i^c + \alpha_i a_i^{\Delta}$, $\alpha_i \in \langle -1, 1\rangle$. The condition (24) holds iff for a certain $\boldsymbol{\alpha} \in \langle -1, 1\rangle^n$ we have

$$
\begin{aligned}
\mathbf{a}^c\mathbf{x} + \sum_{i=1}^n \alpha_i a_i^{\Delta} x_i + \mathbf{b}^c\mathbf{z} &\leq \overline{\beta} + \mathbf{b}^{\Delta}|\mathbf{z}|,\\
\mathbf{a}^c\mathbf{y} + \sum_{i=1}^n \alpha_i a_i^{\Delta} y_i + \mathbf{c}^c\mathbf{z} &\leq \overline{\gamma} + \mathbf{c}^{\Delta}|\mathbf{z}|
\end{aligned}
$$

or, equivalently, iff the following problem

$$\max\left\{\mathbf{0}^T\boldsymbol{\alpha};\ \sum_{i=1}^n \alpha_i a_i^{\Delta} x_i \leq r_1,\ \sum_{i=1}^n \alpha_i a_i^{\Delta} y_i \leq r_2,\ \boldsymbol{\alpha} \leq \mathbf{e},\ -\boldsymbol{\alpha} \leq \mathbf{e}\right\},$$

where $r_1 \equiv \overline{\beta} - \mathbf{a}^c\mathbf{x} - \mathbf{b}^c\mathbf{z} + \mathbf{b}^{\Delta}|\mathbf{z}|$, $r_2 \equiv \overline{\gamma} - \mathbf{a}^c\mathbf{y} - \mathbf{c}^c\mathbf{z} + \mathbf{c}^{\Delta}|\mathbf{z}|$, has an optimal solution. From the duality theory in linear programming this problem has an optimal solution iff the same holds for the problem

$$
\begin{aligned}
\min\Bigg\{ & r_1u_1 + r_2u_2 + \sum_{i=1}^n (v_i + w_i);\\
& a_i^{\Delta} x_i u_1 + a_i^{\Delta} y_i u_2 + v_i - w_i = 0, \quad u_1, u_2, v_i, w_i \geq 0\ \ \forall i = 1,\dots,n \Bigg\}.
\end{aligned}
$$

For optimal solution v_i, w_i the relation $v_i + w_i = |a_i^{\Delta} x_i u_1 + a_i^{\Delta} y_i u_2|$ holds. Hence we can the linear programming problem rewrite as

$$\min\left\{r_1u_1 + r_2u_2 + \sum_{i=1}^n |a_i^{\Delta} x_i u_1 + a_i^{\Delta} y_i u_2|;\ u_1, u_2 \geq 0\right\}.$$

Since the objective function $f(u_1, u_2) = r_1u_1 + r_2u_2 + \sum_{i=1}^n |a_i^{\Delta} x_i u_1 + a_i^{\Delta} y_i u_2|$ is positive homogeneous, it is sufficient (similarly as in the proof of Lemma 1) to check its nonnegativity only for special points:

(i) If $u_1 = 0,\ u_2 = 1$, then $f(0, 1) \geq 0$ is equal to $r_2 + \mathbf{a}^{\Delta}|\mathbf{y}| \geq 0$, which is the l-th inequality from the system (22).

(ii) Let $u_1 = 1,\ u_2 \geq 0$. The function $f(1, u_2)$ represents a broken line with breaks in $u_2 = 0$ and in $u_2 = -\frac{x_k}{y_k} \geq 0,\ y_k \neq 0$. For the first case the condition $f(1, 0) \geq 0$ is equal to $r_1 + \mathbf{a}^\Delta|\mathbf{x}| \geq 0$, which is the l-th inequality from the system (21). In the second case, for the breaks of the objective function $f(1, u_2)$ we obtain the following inequality

$$r_1 + r_2 \frac{-x_k}{y_k} + \sum_{i=1}^{n} a_i^\Delta \left| x_i - \frac{x_k}{y_k} y_i \right| \geq 0 \quad \forall k = 1, \ldots, n : x_k y_k \leq 0,\ y_k \neq 0.$$

Since $-\frac{x_k}{y_k} = \left| -\frac{x_k}{y_k} \right|$, the inequality is equal to (w.l.o.g. assume $x_k, y_k \neq 0$, for otherwise we get redundant condition)

$$r_1|y_k| + r_2|x_k| + \mathbf{a}^\Delta|y_k\mathbf{x} - x_k\mathbf{y}| \geq 0, \quad \forall k = 1, \ldots, n : x_k y_k < 0,$$

which is the l-th inequality from the system (23). □

Note that the system (23) from Theorem 2 is empty if $\mathbf{x} = \mathbf{y}$, or if $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$. Hence from Theorem 2 two corollaries directly follow.

Corollary 2. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$, $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$, $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$. *Then for certain* $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B} \in \mathbf{B}^I$, $\mathbf{C} \in \mathbf{C}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$ *vectors* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ *form a solution of the system*

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} &\leq \mathbf{b}, \\ \mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{z} &\leq \mathbf{c} \end{aligned}$$

if and only if $\mathbf{x}$ *is a weak solution of interval system*

$$\begin{aligned} \mathbf{A}^I\mathbf{x} + \mathbf{B}^I\mathbf{z} &\leq \mathbf{b}^I, \\ \mathbf{A}^I\mathbf{x} + \mathbf{C}^I\mathbf{z} &\leq \mathbf{c}^I. \end{aligned}$$

Corollary 3. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$, $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$, $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$. *Then for certain* $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B} \in \mathbf{B}^I$, $\mathbf{C} \in \mathbf{C}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$ *vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ *form a nonnegative solution of the system*

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} &\leq \mathbf{b}, \\ \mathbf{A}\mathbf{y} + \mathbf{C}\mathbf{z} &\leq \mathbf{c} \end{aligned}$$

if and only if $\mathbf{x}$ *is a solution of the system*

$$\begin{aligned} \underline{\mathbf{A}}\mathbf{x} + \underline{\mathbf{B}}\mathbf{z} &\leq \overline{\mathbf{b}}, \\ \underline{\mathbf{A}}\mathbf{y} + \underline{\mathbf{C}}\mathbf{z} &\leq \overline{\mathbf{c}}. \end{aligned}$$

Remark 1. Contrary to the situation in common analysis, in interval analysis it is not generally possible to transform an interval system of equations $\mathbf{A}^I\mathbf{x} = \mathbf{b}^I$ to the interval system of inequalities $\mathbf{A}^I\mathbf{x} \leq \mathbf{b}^I$, $-\mathbf{A}^I\mathbf{x} \leq -\mathbf{b}^I$. However, if the interval system of inequalities is integrated with certain dependence structure, such a transformation is possible. The interval system $\mathbf{A}^I\mathbf{x} = \mathbf{b}^I$ is weakly solvable iff there exist $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{b} \in \mathbf{b}^I$ such that the system $\mathbf{A}\mathbf{x} + \mathbf{b}x_{n+1} \leq \mathbf{0}$, $\mathbf{A}(-\mathbf{x}) + \mathbf{b}(-x_{n+1}) \leq \mathbf{0}$,

$x_{n+1} = -1$ is solvable. From Theorem 2 (with assignment $\mathbf{y} = -\mathbf{x}$, $y_{n+1} = -x_{n+1}$) it follows the solvability of the system

$$\begin{aligned} \mathbf{A}^\Delta|\mathbf{x}| + \mathbf{b}^\Delta|x_{n+1}| &\geq \mathbf{A}^c\mathbf{x} + \mathbf{b}^c x_{n+1}, \\ \mathbf{A}^\Delta|-\mathbf{x}| + \mathbf{b}^\Delta|-x_{n+1}| &\geq -\mathbf{A}^c\mathbf{x} - \mathbf{b}^c x_{n+1}, \\ (-\mathbf{A}^c\mathbf{x} - \mathbf{b}^c x_{n+1})|-x_k| + (\mathbf{A}^c\mathbf{x} + \mathbf{b}^c x_{n+1})|x_k| + \mathbf{A}^\Delta|\mathbf{0}| &\geq \mathbf{0}\ \forall k = 1, \ldots, n+1, \\ x_{n+1} &= -1, \end{aligned}$$

equivalently,

$$\mathbf{A}^\Delta|\mathbf{x}| + \mathbf{b}^\Delta \quad \geq |\mathbf{A}^c\mathbf{x} - \mathbf{b}^c|,$$

which is the statement of Oettli–Prager Theorem on solvability of $\mathbf{A}^I\mathbf{x} = \mathbf{b}^I$.

4 Mixed equalities and inequalities

In previous sections we studied dependence structure for linear interval equations and inequalities, respectively. Now we turn our attention to mixed linear interval equations and inequalities with a dependence structure.

Theorem 3. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$, $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$, $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$. *Then for certain* $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B} \in \mathbf{B}^I$, $\mathbf{C} \in \mathbf{C}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$ *vectors* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ *form a solution of the system*

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} &= \mathbf{b}, & (25) \\ \mathbf{A}\mathbf{y} + \mathbf{C}\mathbf{z} &\leq \mathbf{c} & (26) \end{aligned}$$

if and only if they satisfy the system of inequalities

$$\begin{aligned} \mathbf{A}^\Delta|\mathbf{x}| + \mathbf{B}^\Delta|\mathbf{z}| + \mathbf{b}^\Delta &\geq |\mathbf{r}_1|, & (27) \\ \mathbf{A}^\Delta|\mathbf{y}| + \mathbf{r}_2 &\geq \mathbf{0}, & (28) \\ -\mathbf{r}_1\mathbf{y}^T diag(sgn\, \mathbf{x}) + \mathbf{r}_2|\mathbf{x}|^T + (\mathbf{b}^\Delta + \mathbf{B}^\Delta|\mathbf{z}|)|\mathbf{y}|^T + \mathbf{A}^\Delta|\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T| &\geq \mathbf{0}, & (29) \end{aligned}$$

where $\mathbf{r}_1 \equiv \mathbf{b}^c - \mathbf{A}^c\mathbf{x} - \mathbf{B}^c\mathbf{z}$, $\mathbf{r}_2 \equiv \overline{\mathbf{c}} - \mathbf{A}^c\mathbf{y} - \mathbf{C}^c\mathbf{z} + \mathbf{C}^\Delta|\mathbf{z}|$.

Proof. Denote $\mathbf{a}^I \equiv \mathbf{A}^I_{l,\cdot}$, $\mathbf{b}^I \equiv \mathbf{B}^I_{l,\cdot}$, $\mathbf{c}^I \equiv \mathbf{C}^I_{l,\cdot}$, $\beta^I \equiv \mathbf{b}^I_l$, $\gamma^I \equiv \mathbf{c}^I_l$. Let us consider the l-th equality and inequality in the systems (25) and (26) and denote them by

$$\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{z} = \beta,\ \mathbf{a}\mathbf{y} + \mathbf{c}\mathbf{z} \leq \gamma, \tag{30}$$

where $\mathbf{a} \in \mathbf{a}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$, $\beta \in \beta^I$, $\gamma \in \gamma^I$. Let the vector $\mathbf{a} \in \mathbf{a}^I$ in demand have its i-th component in the form $a_i \equiv a_i^c + \alpha_i a_i^\Delta$, where $\alpha_i \in \langle -1, 1\rangle$. The condition (24) holds iff for a certain $\boldsymbol{\alpha} \in \langle -1, 1\rangle^n$ we have

$$\begin{aligned} \mathbf{a}^c\mathbf{x} + \sum_{i=1}^n \alpha_i a_i^\Delta x_i + \mathbf{b}^c\mathbf{z} &\in \langle \beta^c - \mathbf{b}^\Delta|\mathbf{z}| - \beta^\Delta, \beta^c + \mathbf{b}^\Delta|\mathbf{z}| + \beta^\Delta\rangle, \\ \mathbf{a}^c\mathbf{y} + \sum_{i=1}^n \alpha_i a_i^\Delta y_i + \mathbf{c}^c\mathbf{z} &\leq \overline{\gamma} + \mathbf{c}^\Delta|\mathbf{z}| \end{aligned}$$

or, equivalently iff the following problem

$$\max\Big\{\mathbf{0}^T\boldsymbol{\alpha};\ -\sum_{i=1}^n \alpha_i a_i^\Delta x_i \le -r_1+\beta_1,\ \sum_{i=1}^n \alpha_i a_i^\Delta x_i \le r_1+\beta_1,$$
$$\sum_{i=1}^n \alpha_i a_i^\Delta y_i \le r_2,\ \boldsymbol{\alpha}\le \mathbf{e},\ -\boldsymbol{\alpha}\le\mathbf{e}\Big\}$$

has an optimal solution (for $r_1 \equiv \beta^c - \mathbf{a}^c\mathbf{x} - \mathbf{b}^c\mathbf{z}$, $\beta_1 \equiv \beta^\Delta + \mathbf{b}^\Delta|\mathbf{z}|$, $r_2 \equiv \overline{\gamma} - \mathbf{a}^c\mathbf{y} - \mathbf{c}^c\mathbf{z} + \mathbf{c}^\Delta|\mathbf{z}|$). From the duality theory in linear programming this problem has an optimal solution iff the problem

$$\min\Big\{-(r_1-\beta_1)u_1+(r_1+\beta_1)u_2+r_2u_3+\sum_{i=1}^n(v_i+w_i);$$
$$-a_i^\Delta x_i u_1 + a_i^\Delta x_i u_2 + a_i^\Delta y_i u_3 + v_i - w_i = 0,$$
$$u_1,u_2,u_3,v_i,w_i\ge 0\ \ \forall i=1,\dots,n\Big\}$$

has an optimal solution. After substitution $\tilde{u}_1 \equiv u_2 - u_1$ we can rewrite this problem as

$$\min\Big\{(r_1+\beta_1)\tilde{u}_1+2\beta_1u_1+r_2u_3+\sum_{i=1}^n(v_i+w_i);$$
$$a_i^\Delta x_i\tilde{u}_1 + a_i^\Delta y_i u_3 + v_i - w_i = 0,$$
$$u_1\ge -\tilde{u}_1,\ u_1,u_3,v_i,w_i\ge 0\ \ \forall i=1,\dots,n\Big\}.$$

For optimal solution v_i, w_i, u_1 it must $v_i + w_i = |a_i^\Delta x_i\tilde{u}_1 + a_i^\Delta y_i u_3|$, $u_1 = (-\tilde{u}_1)^+$ hold. Hence the problem is simplified to

$$\min\Big\{(r_1+\beta_1)\tilde{u}_1+2\beta_1(-\tilde{u}_1)^+ + r_2u_3+\sum_{i=1}^n|a_i^\Delta x_i\tilde{u}_1 + a_i^\Delta y_i u_3|;$$
$$\tilde{u}_1\in\mathbb{R},\ u_3\ge 0\Big\}.$$

Since the positive part of a real number p is equal to $p^+ = \frac{1}{2}(p+|p|)$, the linear programming problem has the final form

$$\min\Big\{r_1\tilde{u}_1+r_2u_3+\beta_1|\tilde{u}_1|+\sum_{i=1}^n|a_i^\Delta x_i\tilde{u}_1 + a_i^\Delta y_i u_3|;\ \tilde{u}_1\in\mathbb{R},\ u_3\ge 0\Big\}.$$

The objective function $f(u_1,u_2) = r_1\tilde{u}_1 + r_2u_3 + \beta_1|\tilde{u}_1)| + \sum_{i=1}^n |a_i^\Delta x_i\tilde{u}_1 + a_i^\Delta y_i u_3|$ of this problem is positive homogeneous, thus it is sufficient (similar as in the proof of Lemma 1) to check the nonnegativity only for some special points:

(i) If $\tilde{u}_1 = \pm 1,\ u_3 = 0$, then $f(\pm 1, 0)\ge 0$ is equal to $\pm r_1 + \beta_1 + \mathbf{a}^\Delta|\mathbf{x}| \ge 0$, or equivalently $\beta_1 + \mathbf{a}^\Delta|\mathbf{x}| \ge |r_1|$, which is the l-th inequality from the system(27).

(ii) Let $u_3 = 1$. The function $f(\tilde{u}_1, 1)$ represents a broken line with breaks in $\tilde{u}_1 = 0$ and in $\tilde{u}_1 = -\frac{y_k}{x_k} \ge 0,\ x_k \ne 0$. For the first case the function $f(0,1)\ge 0$

is equal to $r_2 + \mathbf{a}^\Delta|\mathbf{y}| \geq 0$, which is the l-th inequality from the system (28). In the second case, for each nonzero break of the objective function $f(\tilde{u}_1, 1)$ we obtain inequality

$$r_1 \frac{-y_k}{x_k} + r_2 + \beta_1 \left|\frac{-y_k}{x_k}\right| + \sum_{i=1}^{n} a_i^\Delta \left| -\frac{y_k}{x_k} x_i + y_i \right| \geq 0.$$

This inequality can be expressed (since for $x_k = 0$ we get a redundant condition) as

$$-r_1 y_k sgn(x_k) + r_2|x_k| + \beta_1|y_k| + \mathbf{a}^\Delta|y_k\mathbf{x} - x_k\mathbf{y}| \geq 0, \quad \forall k = 1, \dots, n,$$

or in the vector form

$$-r_1\mathbf{y}^T diag(sgn\,\mathbf{x}) + r_2|\mathbf{x}|^T + \beta_1|\mathbf{y}|^T + \mathbf{a}^\Delta|\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T| \geq \mathbf{0},$$

which corresponds to the l-th inequality from the system (29). □

Putting $\mathbf{x} = \mathbf{y}$ we immediately have he following corollary.

Corollary 4. *Let $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$, $\mathbf{B}^I, \mathbf{C}^I \subset \mathbb{R}^{m\times h}$, $\mathbf{b}^I, \mathbf{c}^I \subset \mathbb{R}^m$. Then for certain $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B} \in \mathbf{B}^I$, $\mathbf{C} \in \mathbf{C}^I$, $\mathbf{b} \in \mathbf{b}^I$, $\mathbf{c} \in \mathbf{c}^I$ vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ form a solution of the system*

$$\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b}, \tag{31}$$
$$\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{z} \leq \mathbf{c} \tag{32}$$

if and only if they are a weak solution of the interval system

$$\mathbf{A}^I\mathbf{x} + \mathbf{B}^I\mathbf{z} = \mathbf{b}^I, \tag{33}$$
$$\mathbf{A}^I\mathbf{x} + \mathbf{C}^I\mathbf{z} \leq \mathbf{c}^I, \tag{34}$$
$$\mathbf{C}^I\mathbf{z} - \mathbf{B}^I\mathbf{z} \leq \mathbf{c}^I - \mathbf{b}^I. \tag{35}$$

Proof. According to Theorem 3 vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ form a solution of the system (31)–(32) iff they satisfy the system

$$\begin{aligned} \mathbf{A}^\Delta|\mathbf{x}| + \mathbf{B}^\Delta|\mathbf{z}| + \mathbf{b}^\Delta &\geq |\mathbf{r}_1|, \\ \mathbf{A}^\Delta|\mathbf{x}| + \mathbf{r}_2 &\geq \mathbf{0}, \\ -\mathbf{r}_1\mathbf{x}^T diag(sgn\,\mathbf{x}) + \mathbf{r}_2|\mathbf{x}|^T + (\mathbf{b}^\Delta + \mathbf{B}^\Delta|\mathbf{z}|)|\mathbf{x}|^T + \mathbf{A}^\Delta|\mathbf{x}\mathbf{x}^T - \mathbf{x}\mathbf{x}^T| &\geq \mathbf{0}. \end{aligned}$$

From [3, Theorem 2.9 a 2.19] it follows that the first and second inequalities of this system are equivalent to (33) and (34), respectively. The third inequality can be rewritten as

$$\begin{aligned} -(\mathbf{b}^c - \mathbf{A}^c\mathbf{x} - \mathbf{B}^c\mathbf{z})|\mathbf{x}^T| + (\overline{\mathbf{c}} - \mathbf{A}^c\mathbf{x} - \mathbf{C}^c\mathbf{z} + \mathbf{C}^\Delta|\mathbf{z}|)|\mathbf{x}|^T & \\ +(\mathbf{b}^\Delta + \mathbf{B}^\Delta|\mathbf{z}|)|\mathbf{x}|^T &\geq \mathbf{0}. \end{aligned} \tag{36}$$

If $\mathbf{x} = \mathbf{0}$, then the statement holds. Assume that $\mathbf{x} \neq \mathbf{0}$. Then the inequality (36) can be simplified to

$$-(\mathbf{b}^c - \mathbf{A}^c\mathbf{x} - \mathbf{B}^c\mathbf{z}) + (\overline{\mathbf{c}} - \mathbf{A}^c\mathbf{x} - \mathbf{C}^c\mathbf{z} + \mathbf{C}^\Delta|\mathbf{z}|) + (\mathbf{b}^\Delta + \mathbf{B}^\Delta|\mathbf{z}|) \geq \mathbf{0},$$

and consequently to

$$\mathbf{C}^\Delta|\mathbf{z}| + \mathbf{B}^\Delta|\mathbf{z}| + \overline{\mathbf{c}} - \underline{\mathbf{b}} \geq \mathbf{C}^c\mathbf{z} - \mathbf{B}^c\mathbf{z}.$$

According to [3, Theorem 2.19] this inequality is equivalent to (35). □

Theorem 4. *Let* $\mathbf{A}^I \subset \mathbb{R}^{m\times n}$, $\mathbf{B}_i^I \subset \mathbb{R}^{m\times h}$, $\mathbf{C}_j^I \subset \mathbb{R}^{m\times h}$, $\mathbf{b}_i^I \subset \mathbb{R}^m$, $\mathbf{c}_j^I \subset \mathbb{R}^m$, $i = 1,\ldots,p$, $j = 1,\ldots,q$. *Then for certain matrices* $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B}_i \in \mathbf{B}_i^I$, $\mathbf{C}_j \in \mathbf{C}_j^I$, $\mathbf{b}_i \in \mathbf{b}_i^I$, $\mathbf{c}_j \in \mathbf{c}_j^I$, $i = 1,\ldots,p$, $j = 1,\ldots,q$ *vectors* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^h$ *form a solution of the system*

$$\begin{aligned} \mathbf{Ax} + \mathbf{B}_i\mathbf{z} &= \mathbf{b}_i, \quad \forall i = 1,\ldots,p, && (37)\\ \mathbf{Ax} + \mathbf{C}_j\mathbf{z} &\le \mathbf{c}_j, \quad \forall j = 1,\ldots,q && (38)\end{aligned}$$

if and only if they are a weak solution of the interval system

$$\begin{aligned} \mathbf{A}^I\mathbf{x} + \mathbf{B}_i^I\mathbf{z} &= \mathbf{b}_i^I, \quad \forall i = 1,\ldots,p && (39)\\ \mathbf{A}^I\mathbf{x} + \mathbf{C}_j^I\mathbf{z} &\le \mathbf{c}_j^I, \quad \forall j = 1,\ldots,q && (40)\\ \big(\mathbf{B}_i^I - \mathbf{B}_k^I\big)\mathbf{z} &= \mathbf{b}_i^I - \mathbf{b}_k^I, \quad \forall i,k : i<k. && (41)\\ \big(\mathbf{C}_j^I - \mathbf{B}_i^I\big)\mathbf{z} &\le \mathbf{c}_j^I - \mathbf{b}_i^I, \quad \forall i,j. && (42)\end{aligned}$$

Proof. One implication is obvious. If for certain $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{B}_i \in \mathbf{B}_i^I$, $\mathbf{C}_j \in \mathbf{C}_j^I$, $\mathbf{b}_i \in \mathbf{b}_i^I$, $\mathbf{c}_j \in \mathbf{c}_j^I$ vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^h$ satisfy the system (37)–(38), then they represent a weak solution of the interval system (39)–(42) as well.

To prove the second implication, denote $\mathbf{a}^I \equiv \mathbf{A}^I_{l,\cdot}$, $\mathbf{b}_i^I \equiv (\mathbf{B}_i^I)_{l,\cdot}$, $\mathbf{c}_j^I \equiv (\mathbf{C}_j^I)_{l,\cdot}$, $\beta_i^I \equiv (\mathbf{b}_i^I)_l$, $\gamma_j^I \equiv (\mathbf{c}_j^I)_l$, $i = 1,\ldots,p$, $j = 1,\ldots,q$. Let us consider the l-th inequalities in systems (37)–(38) and denote them by

$$\begin{aligned} \mathbf{ax} + \mathbf{b}_i\mathbf{z} &= \beta_i, \quad i = 1,\ldots,p, && (43)\\ \mathbf{ax} + \mathbf{c}_j\mathbf{z} &\le \gamma_j, \quad j = 1,\ldots,q, && (44)\end{aligned}$$

where $\mathbf{a} \in \mathbf{a}^I$, $\mathbf{b}_i \in \mathbf{b}_i^I$, $\mathbf{c}_j \in \mathbf{c}_j^I$, $\beta_i \in \beta_i^I$, $\gamma_j \in \beta_j^I$. Denote $r_i \equiv \mathbf{a}^c\mathbf{x} + \mathbf{b}_i^c\mathbf{z} - \beta_i^c$, $i = 1,\ldots,p$ and $s_j \equiv \mathbf{a}^c\mathbf{x} + \mathbf{c}_j^c\mathbf{z} - \gamma_j^c$, $j = 1,\ldots,q$. Vectors $\mathbf{x},\mathbf{z}$ satisfy the system (43)–(44) iff there exists $\alpha \in \langle -\mathbf{a}^\Delta\mathbf{x}, \mathbf{a}^\Delta\mathbf{x}\rangle$ for which we have

$$\begin{aligned} |r_i + \alpha| &\le \mathbf{b}_i^\Delta|\mathbf{z}| + \beta_i^\Delta, \quad i = 1,\ldots,p,\\ s_j + \alpha &\le \mathbf{c}_j^\Delta|\mathbf{z}| + \gamma_j^\Delta, \quad j = 1,\ldots,q\end{aligned}$$

or equivalently

$$\begin{aligned} \alpha &\le \mathbf{a}^\Delta\mathbf{x},\\ \alpha &\ge -\mathbf{a}^\Delta\mathbf{x},\\ \alpha &\le -r_i\mathbf{b}_i^\Delta|\mathbf{z}| + \beta_i^\Delta, \quad i = 1,\ldots,p,\\ \alpha &\ge -r_i - \mathbf{b}_i^\Delta|\mathbf{z}| - \beta_i^\Delta, \quad i = 1,\ldots,p,\\ \alpha &\le -s_j + \mathbf{c}_j^\Delta|\mathbf{z}| + \gamma_j^\Delta, \quad j = 1,\ldots,q.\end{aligned}$$

Such a number α exists iff the following four conditions hold

(i) First condition:

$$\begin{aligned} -\mathbf{a}^\Delta\mathbf{x} &\le -r_i + \mathbf{b}_i^\Delta|\mathbf{z}| + \beta_i^\Delta, \quad i = 1,\ldots,p,\\ \mathbf{a}^\Delta\mathbf{x} &\ge -r_i - \mathbf{b}_i^\Delta|\mathbf{z}| - \beta_i^\Delta, \quad i = 1,\ldots,p,\end{aligned}$$

or, equivalently

$$|r_i| \le \mathbf{b}_i^\Delta|\mathbf{z}| + \beta_i^\Delta, \quad i = 1,\ldots,p.$$

According to [3, Theorem 2.9] the first condition is equivalent to the condition that vectors $\mathbf{x}, \mathbf{z}$ represent a weak solution of the interval equation

$$\mathbf{a}^I\mathbf{x} + \mathbf{b}_i^I\mathbf{z} = \beta_i^I,$$

which corresponds to the l-th equation in the system (39).

(ii) Second condition:

$$-\mathbf{a}^\Delta\mathbf{x} \leq -s_j + \mathbf{c}_j^\Delta|\mathbf{z}| + \gamma_j^\Delta, \quad j = 1, \ldots, q.$$

According to [3, Theorem 2.19] the second condition is equivalent to the condition that vectors $\mathbf{x}, \mathbf{z}$ represent a weak solution of the interval inequality

$$\mathbf{a}^I\mathbf{x} + \mathbf{c}_j^I\mathbf{z} \leq \gamma_j^I,$$

which corresponds to the l-th inequality in the system (40).

(iii) Third condition:

$$\begin{array}{rcl} -r_k - \mathbf{b}_k^\Delta|\mathbf{z}| - \beta_k^\Delta & \leq & -r_i + \mathbf{b}_i^\Delta|\mathbf{z}| + \beta_i^\Delta, \quad i,k = 1,\ldots,p,\ i < k, \\ -r_i - \mathbf{b}_i^\Delta|\mathbf{z}| - \beta_i^\Delta & \geq & -r_k + \mathbf{b}_k^\Delta|\mathbf{z}| + \beta_k^\Delta, \quad i,k = 1,\ldots,p,\ i < k, \end{array}$$

or, equivalently

$$|r_i - r_j| \leq (\mathbf{b}_i^\Delta + \mathbf{b}_k^\Delta)|\mathbf{z}| + \beta_i^\Delta + \beta_k^\Delta, \quad i,k = 1,\ldots,p,\ i < k.$$

According to [3, Theorem 2.9] the third condition is equivalent to the condition that vectors $\mathbf{x}, \mathbf{z}$ represent a weak solution of the interval equation

$$(\mathbf{b}_i^I - \mathbf{b}_k^I)\mathbf{z} = \beta_i^I - \beta_k^I, \quad i,k = 1,\ldots,p,\ i < k,$$

which corresponds to the l-th equation in the system (41).

(iv) Fourth condition:

$$-r_i - \mathbf{b}_i^\Delta|\mathbf{z}| - \beta_i^\Delta \leq -s_j + \mathbf{c}_j^\Delta|\mathbf{z}| + \gamma_j^\Delta, \quad i = 1,\ldots,p,\ j = 1,\ldots,q,$$

or, equivalently

$$s_j - r_i \leq (\mathbf{b}_i^\Delta + \mathbf{c}_j^\Delta)|\mathbf{z}| + \beta_i^\Delta + \gamma_j^\Delta, \quad i = 1,\ldots,p,\ j = 1,\ldots,q.$$

According to [3, Theorem 2.19] the fourth condition is equivalent to the condition that vectors $\mathbf{x}, \mathbf{z}$ represent a weak solution of the interval inequality

$$(\mathbf{c}_j^I - \mathbf{b}_i^I)\mathbf{z} \leq \gamma_j^I - \beta_i^I, \quad i = 1,\ldots,p,\ j = 1,\ldots,q,$$

which corresponds to the l-th inequality in the system (42). □

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