Computing the tolerance in multiobjective linear programming

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Introduction

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$$\max_{x \in \mathcal{M}} \; Cx,$$

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$$\mathcal{M} := \{x \in \mathbb{R}^n : Ax \leq b\}.$$

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- *Additive tolerance:* any δ such that x^* remains efficient for all $\hat{C} : |C_{ij} - \hat{C}_{ij}| < \delta$;

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- *Additive tolerance:* any δ such that x^* remains efficient for all $\hat{C} : |C_{ij} - \hat{C}_{ij}| < \delta$;
- *Multiplicative tolerance:* any δ such that x^* remains efficient for all $\hat{C} : |C - \hat{C}| < \delta|G|$.

Computing the tolerances

Normal cone of $\mathcal{M}$ at the point x^*

$$\mathcal{N}(x^*) := \{x \in \mathbb{R}^n : Dx \geq 0\}.$$

For a nondegenerate basic solution, $D = (A_B^{-1})^T$.

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Theorem (Additive tolerance, Hladík 2007)

Let (λ^*, δ^*) be an optimal solution to the linear program

$$\max \delta \quad \textit{subject to} \quad DC^T\lambda - |D|e\delta \geq 0, \;\; \lambda, \delta \geq 0, \;\; e^T\lambda = 1.$$

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Theorem (Multiplicative tolerance, Hladík 2007)

Let (λ^*, δ^*) be an optimal solution to the generalized linear fractional program

$$\max \delta \quad \textit{subject to} \quad DC^T\lambda - \delta|D||G|^T\lambda \geq 0, \ \lambda, \delta \geq 0, \ e^T\lambda = 1.$$

Then δ^* is a multiplicative tolerance.

Example

Example (Hansen, Labbé, Wendell 1989)

Consider

$$C = \begin{pmatrix} 0 & 10 & 0 & 80 \\ 0 & 10 & 10 & 20 \\ 10 & 10 & 10 & 10 \end{pmatrix},$$

$$\begin{aligned} \mathcal{M} = \{x \in \mathbb{R}^4 : \; 4x_1 + 9x_2 + 7x_3 + 10x_4 &\leq 6000, \\ x_1 + x_2 + 3x_3 + 40x_4 &\leq 4000, \\ x &\geq 0\}, \end{aligned}$$

and the basic solution $x^* \simeq (1333.33, 0, 0, 66.67)^T$.

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- The linear program yields
  $\lambda^* \simeq (0.2343, 0, 0.7657)$, $\delta^* \simeq 2.0749$.
- The generalized linear fractional program (with $G = C$) yields
  $\lambda^* \simeq (0.2665, 0, 0.7335)$, $\delta^* \simeq 0.2195$.

Additive tolerance matrix

Compute upper matrix Δ^+ and lower matrix Δ^- such that x^* detains efficiency for every $\hat{C}$ with $-\Delta^- < \hat{C} - C < \Delta^+$.

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Algorithm

Let (λ^*, δ^*) be an optimal solution to the linear program.

1. *For each $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, n\}$ set*

$$\Delta_{ij}^+ := \inf_{k\colon D_{kj}<0} \frac{D_{k\cdot} C^T \lambda^*}{|D|_{k\cdot} e},$$

$$\Delta_{ij}^- := \inf_{k\colon D_{kj}>0} \frac{D_{k\cdot} C^T \lambda^*}{|D|_{k\cdot} e}.$$

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$$\Delta^+_{ij} := \infty \ \forall j \in \{1, \ldots, n\},$$

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Proof and example

Sketch of proof.

The possible deviations $\hat{C}$ of C are determined so that the inequality $D\hat{C}^T\lambda^* > 0$ remains true. $\square$

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Reconsider our problem. The first step of Algorithm results in

$$\Delta^+ = \begin{pmatrix} \delta^* & \delta^* & 2.2165 & \delta^* \\ \delta^* & \delta^* & 2.2165 & \delta^* \\ \delta^* & \delta^* & 2.2165 & \delta^* \end{pmatrix}, \quad \Delta^- = \begin{pmatrix} \delta^* & \infty & \infty & \delta^* \\ \delta^* & \infty & \infty & \delta^* \\ \delta^* & \infty & \infty & \delta^* \end{pmatrix},$$

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where $\delta^* = 2.0749$. The second step yields

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$$\Delta^+ = \begin{pmatrix} \delta^* & \delta^* & 0.2849 & \delta^* \\ \delta^* & \delta^* & 0.2849 & \delta^* \\ \delta^* & \delta^* & 0.2849 & \delta^* \end{pmatrix}, \quad \Delta^- = \begin{pmatrix} \delta^* & \infty & \infty & \delta^* \\ \delta^* & \infty & \infty & \delta^* \\ \delta^* & \infty & \infty & \delta^* \end{pmatrix},$$

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Application to interval MOLP

An interval MOLP problem

$$\max_{x \in \mathcal{M}} \; Cx,$$

where C varies in $C^I = \{C \subseteq \mathbb{R}^{s \times n} : \underline{C} \leq C \leq \overline{C}\}$.

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Definition

- A vector $x \in \mathcal{M}$ is possibly efficient if it is efficient for some $C \in C^I$.
- A vector $x \in \mathcal{M}$ is necessarily efficient if it is efficient for every $C \in C^I$.

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Let us denote $C^c := \frac{1}{2}.(\overline{C} + \underline{C})$ and consider the MOLP problem

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$$\max_{x \in \mathcal{M}} \; C^c x.$$

Let (λ^*, δ^*) be an optimal solution to the linear (or generalized linear fractional) program.

Sufficient condition

Theorem

The vector x^* is necessarily efficient if $DC^I\lambda^* > 0$.

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Consider the interval matrix

$$C^I = \begin{pmatrix} [-1,1] & [8,12] & [-1,1] & [75,85] \\ [-1,1] & [8,12] & [8,12] & [17,23] \\ [8,12] & [8,12] & [8,12] & [8,12] \end{pmatrix}$$

The feasible set $\mathcal{M}$ and its point x^* is defined as before.

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$$C^c = \begin{pmatrix} 0 & 10 & 0 & 80 \\ 0 & 10 & 10 & 20 \\ 10 & 10 & 10 & 10 \end{pmatrix}.$$

Then $\lambda^* \simeq (0.2343, 0, 0.7657)$ (or $\lambda^* \simeq (0.2665, 0, 0.7335)$), $DC^I\lambda^* > 0$ and hence x^* is necessarily efficient.

The End.