Separation of convex polyhedral sets with uncertain data

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Separation of polyhedral sets

Introduce two convex polyhedral sets ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, $\mathbf{b} \in \mathbb{R}^{m}$, $\mathbf{d} \in \mathbb{R}^{l}$):

$$\mathcal{M}_1 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}, \quad (1)$$

$$\mathcal{M}_2 \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \le \mathbf{d}\}. \quad (2)$$

- Weak separation of $\mathcal{M}_1$, $\mathcal{M}_2$

  exists a hyperplane $\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} = s\}$ such that $\mathcal{M}_1 \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \le s\}$, $\mathcal{M}_2 \subseteq \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \ge s\}$.

- Strong separation of $\mathcal{M}_1$, $\mathcal{M}_2$

  $\mathcal{M}_1, \mathcal{M}_2$ are weak separable and $dim\, \mathcal{M}_1 = dim\, \mathcal{M}_2 = n$.

Theorem 1. *Convex sets $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{R}^n$ are strongly separable if and only if $dim\, \mathcal{M}_1 = dim\, \mathcal{M}_2 = n$, and $int\, \mathcal{M}_1 \cap int\, \mathcal{M}_2 = \emptyset$.*

Separation theorems

$$\mathcal{Q}^* \equiv \left\{ (\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathbb{R}_+^{m+l+1} \mid \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T & \mathbf{0} \\ \mathbf{b}^T & \mathbf{d}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Theorem 2. *Suppose that* $dim\, \mathcal{M}_1 = dim\, \mathcal{M}_2 = n,\ int\, \mathcal{M}_1 \cap int\, \mathcal{M}_2 = \emptyset$*. Let* $(\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*,\ \mathbf{u}^T \mathbf{A} \neq \mathbf{0}^T$*, and* $\eta \in \langle 0, v_{l+1} \rangle$ *is arbitrary. Then*

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T (\mathbf{A}\mathbf{x} - \mathbf{b}) = \eta\} \tag{3}$$

represents a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1,\ \mathcal{M}_2$*. Conversely, any separating hyperplane* $\mathcal{R}$ *of* $\mathcal{M}_1,\ \mathcal{M}_2$ *can be expressed in the form of* (3) *for a certain* $(\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}^*,\ \mathbf{u}^T \mathbf{A} \neq \mathbf{0}^T$*, and* $\eta \in \langle 0, v_{l+1} \rangle$*.*

Theorem 3. *Let* $dim\, \mathcal{M}_1 = dim\, \mathcal{M}_2 = n$*. Then the convex sets* $\mathcal{M}_1$*,* $\mathcal{M}_2$ *are strongly separable if and only if* $\mathcal{Q}^* \neq \emptyset$*.*

Interval matrices, iquations and inequalities

Interval matrix is defined as $\mathbf{M}^I = \{\mathbf{M} \in \mathbb{R}^{m\times n} \mid \underline{\mathbf{M}} \le \mathbf{M} \le \overline{\mathbf{M}}\}$. Next introduce

$$\mathbf{M}^c \equiv \frac{1}{2}.(\overline{\mathbf{M}} + \underline{\mathbf{M}}), \quad \mathbf{M}^\Delta \equiv \frac{1}{2}.(\overline{\mathbf{M}} - \underline{\mathbf{M}}).$$

The system of interval linear inequalities

$$\mathbf{M}^I\mathbf{x} \le \mathbf{m}^I$$

- is strongly solvable, if $\mathbf{M}\mathbf{x} \le \mathbf{m}$ is solvable for all $\mathbf{M} \in \mathbf{M}^I$, $\mathbf{m} \in \mathbf{m}^I$.
  A vector $\mathbf{x}^0$ is a strong solution, if $\mathbf{M}\mathbf{x}^0 \le \mathbf{m}$ holds for all $\mathbf{M} \in \mathbf{M}^I$, $\mathbf{m} \in \mathbf{m}^I$.
- is weakly solvable, if $\mathbf{M}\mathbf{x}^0 \le \mathbf{m}$ holds for a certain vector $\mathbf{x}^1$ and $\mathbf{M} \in \mathbf{M}^I$, $\mathbf{m} \in \mathbf{m}^I$ (such a vector $\mathbf{x}^0$ is called a weak solution).

Separation of interval convex polyhedral sets

Let us consider two families of convex polyhedral sets

$$\mathcal{M}_1^I \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}^I\mathbf{x} \leq \mathbf{b}^I\}, \tag{4}$$

$$\mathcal{M}_2^I \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}^I\mathbf{x} \leq \mathbf{d}^I\}, \tag{5}$$

Assuptions for $\mathcal{M}_1^I$ (similarly for $\mathcal{M}_2^I$):

- no matrix $\mathbf{A} \in \mathbf{A}^I$ contain the zero row
- $dim\, \mathcal{M}_1 = n$ holds for all $\mathcal{M}_1 \in \mathcal{M}_1^I$
  (i.e., the interval system $\mathbf{A}^I\mathbf{x} < \mathbf{b}^I$ is strongly solvable or, equivalently, the system $\overline{\mathbf{A}}\mathbf{x}^1 - \underline{\mathbf{A}}\mathbf{x}^2 < \underline{\mathbf{b}},\ \mathbf{x}^1, \mathbf{x}^2 \geq \mathbf{0}$ is solvable)

Separability for some realization

According to Theorem 3, there exists two convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ which are strongly separable if and only if the interval system

$$\begin{pmatrix} (\mathbf{A}^I)^T & (\mathbf{C}^I)^T & \mathbf{0} \\ (\mathbf{b}^I)^T & (\mathbf{d}^I)^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} \geq \mathbf{0}. \qquad (6)$$

is weakly solvable. From Oettli-Prager Theorem we have that interval system (6) is weakly solvable if and only if the system

$$\begin{pmatrix} \underline{\mathbf{A}}^T & \underline{\mathbf{C}}^T & \mathbf{0} \\ \underline{\mathbf{b}}^T & \underline{\mathbf{d}}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} \leq \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix} \leq \begin{pmatrix} \overline{\mathbf{A}}^T & \overline{\mathbf{C}}^T & \mathbf{0} \\ \overline{\mathbf{b}}^T & \overline{\mathbf{d}}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix}, \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} \geq \mathbf{0}$$

is solvable.

Separability for all realizations

To verify whether all convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ are strongly separable, it is equivalent to verify whether interval system (6) is strongly solvable.

Theorem 4. *An interval system $\mathbf{M}^I\mathbf{x} = \mathbf{m}^I$, $\mathbf{x} \geq \mathbf{0}$ is strongly solvable if and only the system*

$$(\mathbf{M}^c - diag(\mathbf{z})\mathbf{M}^\Delta)\mathbf{x} = \mathbf{m}^c + diag(\mathbf{z})\mathbf{m}^\Delta, \quad \mathbf{x} \geq \mathbf{0}$$

is solvable for each $\mathbf{z} \in \{\pm 1\}^m$.

Theorem 5. *Let $\mathbf{N}^I \subset \mathbb{R}^{n \times 2n}$. Checking strong solvability of the interval system*

$$\mathbf{N}^I\mathbf{x} = \mathbf{0},\ \mathbf{x} \gneq \mathbf{0} \tag{7}$$

is an NP-hard problem.

Separability for all realizations (2)

Sufficient condition: Each two convex polyhedral sets $\mathcal{M}_1 \in \mathcal{M}_1^I$, $\mathcal{M}_2 \in \mathcal{M}_2^I$ are strongly separable, if convex hulls $conv\ \left(\cup_{\mathcal{M}_1 \in \mathcal{M}_1^I} \mathcal{M}_1\right)$, $conv\ \left(\cup_{\mathcal{M}_2 \in \mathcal{M}_2^I} \mathcal{M}_2\right)$ are strongly separable. From Gerlach Theorem it follows that a vector $\mathbf{x} \in \mathbb{R}^n$ is a weak solution of $\mathbf{A}^I \mathbf{x} \leq \mathbf{b}^I$ if and only if the vector $\mathbf{x}$ solves

$$(\mathbf{A}^c - \mathbf{A}^\Delta diag(\mathbf{z}))\mathbf{x} \leq \overline{\mathbf{b}}$$

for some $\mathbf{z} \in \{\pm 1\}^n$. Hence

$$\bigcup_{\mathcal{M}_1 \in \mathcal{M}_1^I} \mathcal{M}_1 = \bigcup_{\mathbf{z} \in \{\pm 1\}^n} \left\{ \mathbf{x} \in \mathbb{R}^n \mid (\mathbf{A}^c - \mathbf{A}^\Delta diag(\mathbf{z}))\mathbf{x} \leq \overline{\mathbf{b}} \right\}.$$

Separation of interval convex polytopes

Convex polytopes $\mathcal{M}_1$, $\mathcal{M}_2$ are described by the lists of their vertices as follows

$$\mathcal{M}_1 \text{ has vertices } \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n,\ m \geq 1,$$

$$\mathcal{M}_2 \text{ has vertices } \mathbf{c}_1, \ldots, \mathbf{c}_l \in \mathbb{R}^n,\ l \geq 1.$$

Denote as $\mathbf{A} \in \mathbb{R}^{m \times n}$ such a matrix, for which $\mathbf{A}_{i,\cdot} = \mathbf{a}_i^T$, $i \in \{1, \ldots, m\}$ and analogously $\mathbf{C} \in \mathbb{R}^{l \times n}$ is such a matrix for which $\mathbf{C}_{j,\cdot} = \mathbf{c}_j^T$, $j \in \{1, \ldots, l\}$. We will also use the more transparent notation $\mathcal{M}_1 \equiv \mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2 \equiv \mathcal{M}_2(\mathbf{C})$.

The convex polytopes $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ are weakly separable if and only if

$$\left\{ (\mathbf{r}, s) \in \mathbb{R}^{n+1} \mid \begin{pmatrix} \mathbf{A} & -\mathbf{1} \\ -\mathbf{C} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ s \end{pmatrix} \leq \mathbf{0},\ \mathbf{r} \neq \mathbf{0} \right\} \neq \emptyset.$$

For interval analysis assume that $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$.

Separability for some realization

The convex polytopes $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ are weakly separable for some $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$ if and only if the interval system

$$\begin{pmatrix} \mathbf{A}^I & -\mathbf{1} \\ -\mathbf{C}^I & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ s \end{pmatrix} \leq \mathbf{0}, \ \mathbf{r} \neq \mathbf{0} \tag{8}$$

is weakly solvable. But according to Theorem 6 this is NP-hard problem.

Theorem 6. *Given an interval matrix* $\mathbf{M}^I \subset \mathbb{R}^{m \times n}$. *Checking weak solvability of an interval system*

$$\mathbf{M}^I \mathbf{x} \leq \mathbf{0}, \ \mathbf{x} \neq \mathbf{0}$$

is NP-hard problem.

Separability for some realization (2)

For checking (with exponential complexity) weak solvability of (8) we can use Theorem 7.

Theorem 7. *An interval system* $\mathbf{M}^I\mathbf{x} \le \mathbf{0},\ \mathbf{x} \ne \mathbf{0}$ *is weakly solvable if and only if the system*

$$(\mathbf{M}^c - \mathbf{M}^\Delta diag(\mathbf{z}))\mathbf{x} \le \mathbf{0},\ \mathbf{x} \ne \mathbf{0}$$

is solvable for some $\mathbf{z} \in \{\pm 1\}^n$*.*

Separability for all realizations

The convex polytopes $\mathcal{M}_1(\mathbf{A})$, $\mathcal{M}_2(\mathbf{C})$ are weakly separable for all $\mathbf{A} \in \mathbf{A}^I$ and $\mathbf{C} \in \mathbf{C}^I$ if and only if the interval system

$$\begin{pmatrix} \mathbf{A}^I & -\mathbf{1} \\ -\mathbf{C}^I & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ s \end{pmatrix} \leq \mathbf{0},\ \mathbf{r} \neq \mathbf{0} \tag{9}$$

is strongly solvable. If the interval system (9) has a strong solution $(\mathbf{r}, s)$, then simply (9) is strongly solvable and for the hyperplane $\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} = s\}$ we have

$$\mathcal{M}_1(\mathbf{A}) \subset \overline{\mathcal{R}^-} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \leq s\} \quad \forall \mathbf{A} \in \mathbf{A}^I,$$
$$\mathcal{M}_2(\mathbf{C}) \subset \overline{\mathcal{R}^+} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}^T\mathbf{x} \geq s\} \quad \forall \mathbf{C} \in \mathbf{C}^I.$$

Theorem 8. *Let $dim\,\mathcal{M}_1(\mathbf{A}) = dim\,\mathcal{M}_2(\mathbf{C}) = n$ for all $\mathbf{A} \in \mathbf{A}^I$, $\mathbf{C} \in \mathbf{C}^I$. Then the interval system (9) has a strong solution iff (9) is strongly solvable.*

Separability for all realizations (2)

- The interval system (9) has a strong solution if and only if the system

$$\begin{pmatrix} \overline{\mathbf{A}} & -\underline{\mathbf{A}} & -\mathbf{1} \\ -\underline{\mathbf{C}} & \overline{\mathbf{C}} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{r}^1 \\ \mathbf{r}^2 \\ s \end{pmatrix} \le \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{r}^1 \neq \mathbf{r}^2, \quad \mathbf{r}^1, \mathbf{r}^2 \ge \mathbf{0} \qquad (10)$$

  is solvable. Moreover, if $(\mathbf{r}^1, \mathbf{r}^2, s)$ solves (10), then the vector $(\mathbf{r}^1 - \mathbf{r}^2, s)$ is the required strong solution of the interval system (9).

- Compute $\cup_{\mathbf{A}\in\mathbf{A}^I} \mathcal{M}_1(\mathbf{A})$, $\cup_{\mathbf{C}\in\mathbf{C}^I} \mathcal{M}_2(\mathbf{C})$ and check their weak separability. Denoting by $\mathbf{a}_i^j$, $i \in \{1, \ldots, m\}$, $j \in J$ ($|J| = 2^n$), vertices of $\mathbf{A}^I_{i,\cdot}$ we have

$$\bigcup_{\mathbf{A}\in\mathbf{A}^I} \mathcal{M}_1(\mathbf{A}) = conv \left(\bigcup_{i\in\{1,\ldots,m\}} \bigcup_{j\in J} \{\mathbf{a}_i^j\} \right).$$

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