Homeworks from Fundamentals of Nonlinear Optimization

(Milan Hladík, November 7, 2022)

For tutorial credits, it is needed at least 30% of points of each series.

(A) Series: Generalized convex functions **44**

1. Classify the following functions (check if they are convex, quasiconvex, pseudoconvex, quasilinear, concave, ...):
   (a) xy on $\mathbb{R}^2_+$, **4**
   (b) $\frac{1}{xy}$ on $\mathbb{R}^2_+$, **4**
   (c) $\frac{x}{y}$ on $\mathbb{R}^2_+$, **4**
   (d) $\sqrt{x}e^{-x}$ on $\mathbb{R}_+$, **4**

   Find the most precise classification (for example, show that a given function is quasiconvex, but not explicitly quasiconvex).
2. Find an interesting example of a quasiconvex function. **4**
3. Show this characterization of quasiconvexity of a function $f\colon \mathbb{R} \to \mathbb{R}$: There exists $a \in \mathbb{R} \cup \{\pm\infty\}$ such that $f(x)$ is
   - either nonincreasing on $(-\infty, a)$ and nondecreasing on $[a, \infty)$,
   - or nonincreasing on $(-\infty, a]$ and nondecreasing on (a, ∞). **6**
4. Let $M \subseteq \mathbb{R}^n$ be convex and let $f, g\colon M \to \mathbb{R}$. Consider the product $f(x) \cdot g(x)$.
   (a) What we get if f, g are both concave and nonnegative? **4**
   (b) What we get if f, g are both convex and nonnegative? **4**
5. For differentiable functions, compare explicitly quasiconvex and pseudoconvex functions. **6**
6. Decide if the following statement is true: If $f\colon \mathbb{R} \to \mathbb{R}$ is pseudolinear and invertible, then $f^{-1}(x)$ is pseudolinear as well. **4**

(B) Series: Optimality conditions **38**

1. Generalize Karush-Kuhn-Tucker conditions for the optimization problems having also equality constraints

$$\min f(s);\ g(x)_i \leq 0,\ h_j(x) = 0, \quad i = 1, \ldots, m,\ j = 1, \ldots, k.$$

Do it step by step:

(a) Extend the Gordan theorem to the form: *$Ax < 0$, $Bx = 0$ unsolvable $\Leftrightarrow$ $B^T y + A^T z = 0$, $z \geq 0$, $z \neq 0$ solvable.* **4**

(b) Generalize the lemma: *If x^0 is a local minimum, then there is no $d \in \mathbb{R}^n$ such that $d^T \nabla f(x^0) < 0$, $d^T \nabla g_i(x^0) < 0\ \forall i \in I(x^0)$, $d^T \nabla h_j(x^0) = 0\ \forall j = 1, \ldots, k$.* **4**

(c) Generalize the Fritz John conditions: *If x^0 is a local minimum, then there is $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$ a $\nu \in \mathbb{R}^k$ such that*

$$\begin{aligned} (\mu, \lambda) \geq 0,\ (\mu, \lambda) \neq 0, \\ \lambda^T g(x^0) = 0, \\ \mu \nabla f(x^0) + \lambda^T \nabla g(x^0) + \nu^T \nabla h(x^0) = 0. \end{aligned}$$ **4**

(d) Generalize the Karush-Kuhn-Tucker conditions: *Let x^0 be a local minimum, and let the vectors $\nabla g_i(x^0)$, $i \in I(x^0)$, $\nabla h_j(x^0)$, $j = 1, \ldots, k$, be linearly independent. Then there are $\lambda \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^k$ such that*

$$\begin{aligned} \lambda \geq 0, \\ \lambda^T g(x^0) = 0, \\ \nabla f(x^0) + \lambda^T \nabla g(x^0) + \nu^T \nabla h(x^0) = 0. \end{aligned}$$ **4**

2. Prove the observation mentioned in the lecture: $T \subseteq G$. **6**

3. Prove the observation mentioned in the lecture: $int\, G \subseteq int\, T$. **6**

4. By using KKT conditions, solve the problems

(a) max $c^T x;\ x^T A x \leq 1,\ x \in \mathbb{R}^n$, where $c \neq 0$ and A is positive definite. **4**

(b) max $xy;\ x + y^2 \leq 2,\ x, y \geq 0$. (*Hint.* By analysis of the cases.) **6**

(C) Series: Lagrange duality **26**

1. Consider the problem

$$\min\ x^TQx + q^Tx;\ x^TA_ix + a_i^Tx + b_i \le 0,\ i = 1, \dots, m,$$

   where $Q, A_1, \dots, A_m \in \mathbb{R}^{n\times n}$, $q, a_1, \dots, a_m \in \mathbb{R}^n$, $b_1, \dots, b_m \in \mathbb{R}$, and in addition Q is positive definite and A_i are positive semidefinite. Find the Lagrange dual problem and categorize it. **6**

2. Decide about validity of the statements:

   (a) The dual problem to the dual problem is always the primal problem. **2**

   (b) The above statement holds for every convex programming problem. **4**

3. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$. Consider the optimization problem

$$\min_{x\in\mathbb{R}^n}\ \max_{i=1,\dots,m} (A_{i*}x + b_i).$$

   (a) Reformulate the problem as a linear program and find the standard dual problem. **2**

   (b) Find the Lagrange dual problem for the equivalent form

$$\min_{x\in\mathbb{R}^n}\ \max_{i=1,\dots,m} y_i;\ y = Ax + b. \qquad \mathbf{4}$$

   (c) Compare the optimal value (of the primal or dual problem) with the approximate value computed by solving

$$\min_{x\in\mathbb{R}^n}\ \ln(\textstyle\sum_{i=1}^n \exp(A_{i*}x + b_i)). \qquad \mathbf{4}$$

   (d) Compare the optimal value with the approximate value computed by solving the problem with parameter $\gamma > 0$

$$\min_{x\in\mathbb{R}^n}\ \frac{1}{\gamma}\ln(\textstyle\sum_{i=1}^m \exp(\gamma(A_{i*}x + b_i))). \qquad \mathbf{4}$$

(D) Series: Semidefinite programming **32**

1. Formulate the constraints as constraints from a semidefinite program

   (a) $xy \geq 1$, $x, y \geq 0$ **2**

   (b) $x^4 + y^4 \leq 1$ **4**

2. Consider undirected graph $G = (V, E)$, $V = \{1, \ldots, n\}$, with interval weights on edges $[a_{ij}, b_{ij}]$, $(i, j) \in E$, representing uncertain distances between objects $i, j \in V$. Formulate semidefinite program deciding whether there exists points $x_1, \ldots, x_n \in \mathbb{R}^d$ such that the distance between x_i and x_j lies in the interval $[a_{ij}, b_{ij}]$, $(i, j) \in E$. **4**

3. Formulate as semidefinite program the problem of finding an ellipse containing the points $x_1, \ldots, x_n \in \mathbb{R}^d$, and not containing the points $y_1, \ldots, y_m \in \mathbb{R}^d$ such that the ellipse (a) is as round as possible (in some sense), or (b) has minimal sum of the lengths of semiaxes. **4**

4. Formulate as a semidefinite program the problem

$$\min_{x \in \mathbb{R}^n} \max_{i=1,\ldots,m} |\log(a_i^T x) - \log(b_i)|,$$

   which is a linear regression problem with maximum norm after taking the logarithm of the data. **4**

5. Formulate as semidefinite program and its solution the condition that a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ can be expressed as a sum of squares of some polynoms. *Hint:* Cholesky decomposition of the solution of a suitable SDP. **6**

6. Consider 2-SAT problem, where each clause is a disjunction of exactly two literals. Find 0.878-approximation algorithm to maximize the number of simultaneously satisfiable clauses. **8**